Maximum Flow Problems I

AU4606: Network Optimization

AI4702: Network Intelligence and Optimization

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Last few lectures

- Shortest path problems with nonnegative costs
  - Properties of shortest path problems
  - Algorithms for acyclic graphs (pulling and reaching)
  - Dijkstra's algorithm
- Shortest path problems with negative costs
  - Generic label-correcting algorithms
  - Bellman-Ford algorithm
  - Detecting negative cycles
  - All-pair shortest paths

Today

1. Maximum flow problems: formulation
   - What is a maximum flow problem?
   - An application
2. Maximum flow problems: preliminaries
   - Residual graphs
   - s-t cut
3. Augmenting path algorithms
   - Generic augmenting path algorithm
   - Theoretical properties

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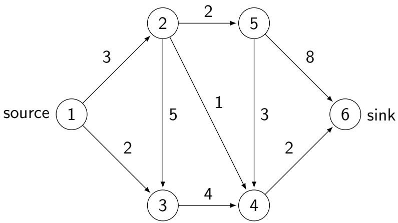
What is a maximum flow problem?



Given

- A directed graph $G = (N, A)$
- Each arc $(i, j) \in A$ has a capacity constraint u_{ij}
- A source node $s \in N$ and a sink node $t \in N$

Define

- A vector of flow variables f_{ij} over arcs such that
  1. They are nonnegative
  2. They are upper bounded by the flow capacity
  3. They satisfy the flow balance equation at nodes other than s and t

Maximize

- The net flow out of node s (the net flow into node t)

What is a maximum flow problem?

$$\begin{aligned} \text{maximize} \quad & v \\ \text{subject to} \quad & \sum_{j:(i,j)\in A} f_{ij} - \sum_{j:(j,i)\in A} f_{ji} = \begin{cases} v, & \text{for } i = s, \\ 0, & \text{for } i \in N \setminus \{s,t\}, \\ -v, & \text{for } i = t, \end{cases} \\ & 0 \le f_{ij} \le u_{ij} \quad \text{for each } (i,j) \in A. \end{aligned}$$

- A vector $f = \{f_{ij}\}$ satisfying constraints is called a (feasible) *flow*, with associated flow value v

Shortest path problems and maximum flow problems

Similarities

- Pervasive in practical applications
- Arise as subproblems for minimum cost flow problems

Differences

- SSPs model arc costs but no capacities
- Maximum flows model capacities but no arc costs

Key concepts introduced in SSPs will be useful in maximum flow problems

- Distance labels
- Optimality conditions

Types of maximum flow algorithms and assumptions

Two different types of algorithms

1. Augmenting path algorithms where balance constraints are maintained
2. Push-relabel algorithms where some nodes have excesses

Types of maximum flow algorithms and assumptions

Two different types of algorithms

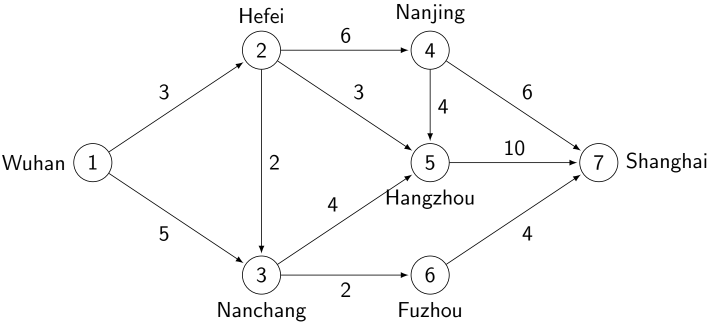
1. Augmenting path algorithms where balance constraints are maintained
2. Push-relabel algorithms where some nodes have excesses

Assumptions

- The graph is directed
- All capacities are nonnegative integers
- The graph does not contain a directed path from s to t composed only of infinite capacity arcs
- The graph does not contain parallel arcs (with same heads and tails)

An application

How many people can get back to Shanghai from Wuhan?



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Residual graphs

Suppose arc (i, j) carries $0 \leq f_{ij} \leq u_{ij}$ units of flow

- The arc can carry additional $u_{ij} - f_{ij}$ units of flow
- A "virtual" f_{ij} units of flow can be sent in the opposite direction

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The residual graph $G(f) = (N', A')$ with respect to a feasible flow f in $G = (N, A)$ is defined as

- $N' = N$
- For each arc $(i,j) \in A$
  1. $(i,j) \in A'$ and it has residual capacity $r_{ij} = u_{ij} - f_{ij}$ if $u_{ij} - f_{ij} > 0$
  2. $(j,i) \in A'$ and it has residual capacity $r_{ji} = f_{ij}$ if $f_{ij} > 0$

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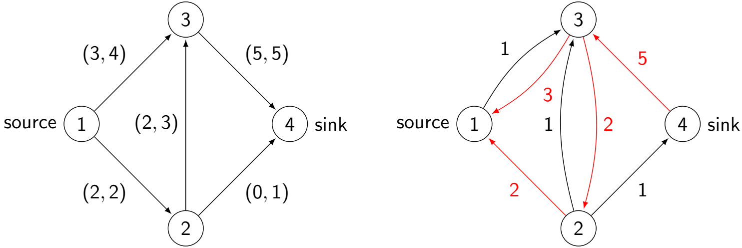
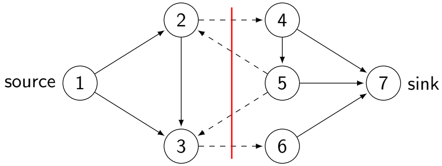


(a) Graph G with flow f (b) Residual graph $G(f)$

- A cut $[S, \bar{S}]$ is a partition of node set N into two subsets S and $\bar{S}$
- A cut $[S, \bar{S}]$ is an s-t cut if $s \in S$ and $t \in \bar{S}$



A cut $S = \{1, 2, 3\}$, $\bar{S} = \{4, 5, 6, 7\}$

- A set of cut arcs $(S, \bar{S}) = \{(i, j) \in A \mid i \in S, j \in \bar{S}\}$
  - In the above example, $(S, \bar{S}) = \{(2, 4), (3, 6)\}$
- Capacity of an s-t cut (directional)

$$u[S, \bar{S}] = \sum_{(i,j) \in (S, \bar{S})} u_{ij}$$

Minimum s-t cut

- A minimum s-t cut is an s-t cut whose capacity is minimum

Value of flow and capacity of cuts

The value of any flow is less than or equal to the capacity of any cut in the graph.

- The maximum flow is less than or equal to the minimum capacity
- If a flow f has value equal to some s-t cut, then f must be maximum

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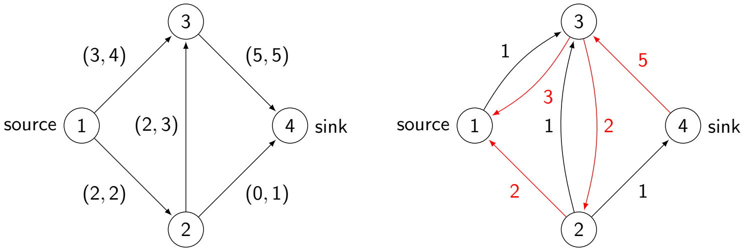
Augmenting paths

- Augmenting path: directed path from s to t in residual graph
- Residual capacity of augmenting path: minimum arc residual capacity



(a) Graph G with flow f (b) Residual graph $G(f)$

- $1 - 3 - 2 - 4$ is an augmenting path
- The residual capacity of $1 - 3 - 2 - 4$ is $\min\{1, 2, 1\} = 1$

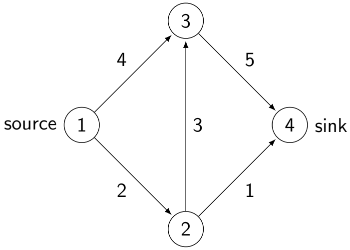
Each augmenting path is a feasible path to send additional flow.

Generic augmenting path algorithm

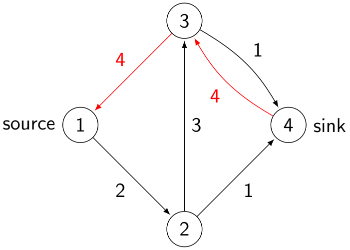
Algorithm Augmenting path

$f \leftarrow 0$
while $G(f)$ contains a directed path from s to t **do**
    Identify an augmenting path P from s to t
    $\delta(P) = \min\{r_{ij}, (i,j) \in P\}$
    Augment $\delta(P)$ units of flow along P and update $G(f)$
end while

Generic augmenting path algorithm: running example



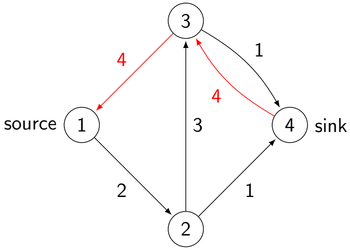
(a) A maximum flow problem



(b) Residual graph 1

An augmenting path $1 - 3 - 4$ exists and 4 units flow can be sent

Generic augmenting path algorithm: running example



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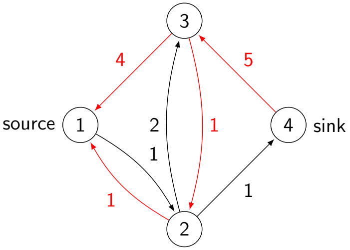
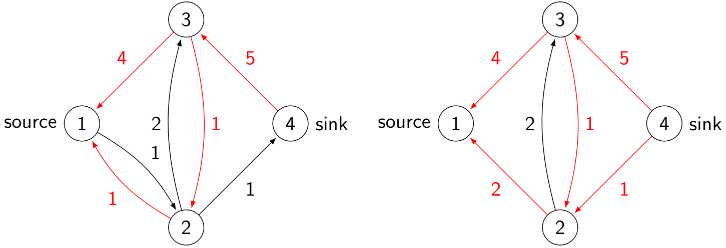


(b) Residual graph 2

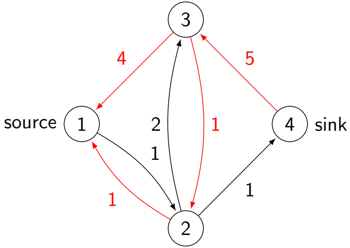
An augmenting path $1 - 2 - 3 - 4$ exists and 1 unit flow can be sent



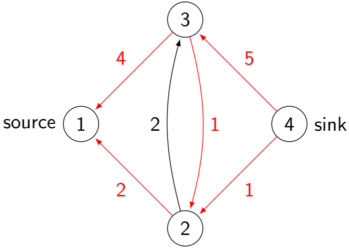
(a) Residual graph 2 (b) Residual graph 3

An augmenting path $1 - 2 - 4$ exists and 1 unit flow can be sent

Generic augmenting path algorithm: running example



(a) Residual graph 2



(b) Residual graph 3

An augmenting path $1 - 2 - 4$ exists and 1 unit flow can be sent

No directed paths exist from s to t in the residual graph, terminate

Correctness of generic augmenting path algorithm

Correctness

If the generic augmenting path algorithm terminates, then the current flow f is a maximum flow.

Augmenting path alg. solves maximum flow problem in $O(m^2U)$.

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Augmenting path alg. solves maximum flow problem in $O(m^2U)$.

Augmenting path theorem

A flow f^* is a maximum flow if and only if the residual graph $G(f^*)$ contains no augmenting path.

Integrality theorem

If all arc capacities are integer, the maximum flow problem has an integer maximum flow.

Maximum flow minimum cut theorem

The maximum value of the flow from a source node s to a sink node t in a capacitated graph equals the minimum capacity among all s-t cuts.

Upcoming

Week 1-8 (AU4606 & AI4702):

- Introduction (1 lecture)
- Preparations (3 lectures)
    - basics of graph theory
    - algorithm complexity and data structure
    - graph search algorithm
- Shortest path problems (3 lectures)
- Maximum flow problems (this and next few lectures)
- Minimum cost flow problems (2 lectures)
- Introduction to multi-agent systems (1 lecture)
- Introduction to cloud networks (1 lecture)

Week 9-16 (AU4606):

- Simplex and network simplex methods (2 lectures)
- Global minimum cut problems (3 lectures)
- Minimum spanning tree problems (3 lectures)