Elements of Matrix Theory

AU7036: Introduction to Multi-agent Systems

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Last time

Motivating problems

- Opinion dynamics
- Averaging in wireless sensor networks
- Flocking dynamics
- Distributed parameter estimation

Today

1. Discrete-time linear systems
2. Jordan normal form
3. Row-stochastic matrices and their spectral radius
4. Nonnegative matrices and Perron-Frobenius theorem

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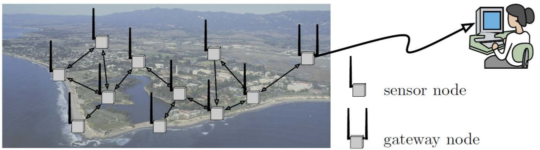
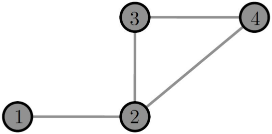
Averaging algorithms in wireless sensor networks





- Suppose each sensor i has initial measurement $x_i(0)$
- Averaging protocol

$$
\begin{aligned}
x_1(k+1) &= \frac{1}{2}(x_1(k) + x_2(k)) \\
x_2(k+1) &= \frac{1}{4}(x_1(k) + x_2(k) + x_3(k) + x_4(k)) \\
x_3(k+1) &= \frac{1}{3}(x_2(k) + x_3(k) + x_4(k)) \\
x_4(k+1) &= \frac{1}{3}(x_2(k) + x_3(k) + x_4(k))
\end{aligned}
$$

- Questions of interest
  - (Stability) Does the iteration converge? Conditions for convergence?
  - (Equilibrium) Where does it converge to?

Discrete-time linear systems

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

Discrete-time linear system

A square matrix $A \in \mathbb{R}^{n\times n}$ defines a discrete-time linear system by

$$x(k+1) = Ax(k), \qquad x(0) = x_0.$$

Solutions to discrete-time linear systems

$$x(k+1) = Ax(k), \qquad x(0) = x_0.$$

$$\Updownarrow$$

$$x(k) = A^k x_0$$

The asymptotic behavior depends on A^k

Semi-convergent and convergent matrices

A matrix $A \in \mathbb{R}^{n\times n}$ is

- semi-convergent if $\lim\limits_{k\to\infty} A^k$ exists, and
- convergent if it is semi-convergent and $\lim\limits_{k\to\infty} A^k = \mathbb{0}_{n\times n}$

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -0.9 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}$$

Today

Jordan normal form: definitions

Jordan normal form

Each matrix $A \in \mathbb{C}^{n\times n}$ is similar to a block diagonal matrix $J \in \mathbb{C}^{n\times n}$, called the Jordan normal form of A, given by

$$J = \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_m \end{bmatrix}$$

where each block J_i, called a Jordan block, is a square matrix of size j_i and of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

Clearly, $m \leq n$ and $j_1 + \cdots + j_m = n$

Jordan normal form: properties

$$T^{-1}AT = J = \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_m \end{bmatrix}$$

- Smallest Jordan block $[\lambda_i]$
- Diagonalization is a special case with all size one blocks $[\lambda_i]$
- Eigenvalues of Jordan blocks (diagonals) are that of A (similarity)
- Two blocks may have the same eigenvalues, e.g.,

$$\begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{bmatrix}$$

Jordan normal form: an example

$$T^{-1}AT = J = \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_m \end{bmatrix}$$

- Algebraic multi. λ_i: sum of sizes of all blocks having λ_i as eigenvalues
- Geometric multi. λ_i: number of blocks having λ_i as eigenvalues
- Simple eigenvalue: algebraic multi. $=$ geometric multi. $= 1$
- Semisimple eigenvalue: algebraic multi. $=$ geometric multi.

Jordan normal form: an example

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$$\begin{bmatrix} 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{cases} 7 \text{ has algebraic mult. 4 and geometric mult. 2,} \\ \quad \text{so that 7 is neither simple nor semisimple,} \\ 8 \text{ has algebraic and geometric mult. 2, so it is semisimple,} \\ 9 \text{ has algebraic and geometric mult. 1, so it is simple.} \end{cases}$$

Jordan normal form: why useful

$$x(k) = A^k x_0$$

Note that $A = TJT^{-1}$, then

$$A^k = \underbrace{TJT^{-1} \cdot TJT^{-1} \cdot \cdots \cdot TJT^{-1}}_{k \text{ times}} = TJ^kT^{-1} = T \begin{bmatrix} J_1^k & 0 & \cdots & 0 \\ 0 & J_2^k & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m^k \end{bmatrix} T^{-1}$$

For a square matrix A with Jordan blocks J_i, the following are equivalent:

- A is semi-convergent (resp. convergent)
- J is semi-convergent (resp. convergent), and
- Each block J_i is semi-convergent (resp. convergent)

Powers of Jordan blocks:

$$\left[\lambda_i^k\right], \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} \\ 0 & \lambda_i^k \end{bmatrix}, \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} \\ 0 & \lambda_i^k & k\lambda_i^{k-1} \\ 0 & 0 & \lambda_i^k \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \dots & \binom{k}{j_i-1}\lambda_i^{k-j_i+1} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \binom{k}{2}\lambda_i^{k-2} \\ 0 & \dots & 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & \dots & \dots & 0 & \lambda_i^k \end{bmatrix}$$

Note that

$$\lim_{k\to\infty} k^h \lambda^k = \begin{cases} 0, & \text{if } |\lambda| < 1, \\ 1, & \text{if } \lambda = 1 \text{ and } h = 0, \\ \text{non-existent or unbounded}, & \text{if } (|\lambda| = 1 \text{ with } \lambda \neq 1) \text{ or } (|\lambda| > 1) \\ & \quad \text{or } (\lambda = 1 \text{ and } h = 1, 2, \dots). \end{cases}$$

- Block J_i of size 1 is convergent if and only if $|\lambda_i| < 1$
- Block J_i of size 1 is semi-convergent iff $|\lambda_i| < 1$ or $\lambda_i = 1$, and
- Block J_i of size ≥ 1 is semi-convergent/convergent iff $|\lambda_i| < 1$

Convergence of discrete-time linear systems

Spectrum and spectral radius

Given a square matrix A

- The spectrum $\mathrm{spec}(A)$ of A is the set of eigenvalues of A; and
- The spectral radius $\rho(A)$ is the maximum norm of eigenvalues of A

$$\rho(A) = \max\{|\lambda|, \lambda \in \mathrm{spec}(A)\}$$

Thm: Convergence and spectral radius

For a square matrix A

- A is convergent if and only if $\rho(A) < 1$,
- A is semi-convergent but not convergence iff
  1. 1 is a semisimple eigenvalue
  2. All other eigenvalues have magnitudes less than 1

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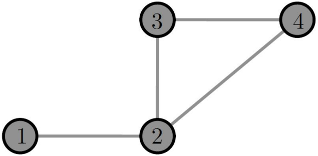
Row-stochastic matrices



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How to determine the spectrum of A? "Obvious" properties of A

- Nonnegative, i.e., $A \geq 0$
- Row sums are one, i.e., $A\mathbb{1}_n = \mathbb{1}_n$

These matrices are called row-stochastic matrices

Spectral radius?

Gershgorin Disk Theorem

For any square matrix $A \in \mathbb{R}^{n \times n}$,

$$\mathrm{spec}(A) \subset \cup_{i=\{1,\cdots,n\}} \{z \| z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|\} \tag{1}$$

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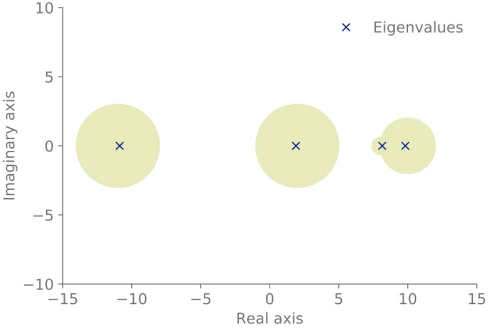


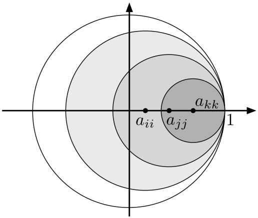
$$\begin{bmatrix} 10 & -1 & 0 & 1 \\ 0.2 & 8 & 0.2 & 0.2 \\ 1 & 1 & 2 & 1 \\ -1 & -1 & -1 & -11 \end{bmatrix}$$

Spectrum of row-stochastic matrices

Row-stochastic matrices

A matrix A is row-stochastic if it satisfies

- nonnegative, i.e., $A \geq 0$
- row sums are one, i.e., $A\mathbb{1}_n = \mathbb{1}_n$



Spectral properties of row-stochastic matrices

If A is row-stochastic, then

- 1 is an eigenvalue
- the spectral radius is 1

Today

Nonnegative matrices

A matrix A is nonnegative if $A \geq 0$.

Two special classes of nonnegative matrices:

Irreducible and primitive matrices

$A \in \mathbb{R}^{n \times n}$, $n \geq 2$ has nonnegative entries and is

- irreducible if $\sum_{k=0}^{n-1} A^k > 0$
- primitive if there exists a positive integer k such that $A^k > 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Primitive matrices are irreducible

If a square matrix A is primitive, then it is also irreducible.

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Cayley-Hamilton Theorem

Let A be an $n \times n$ matrix and $p_A(\lambda) = |\lambda I_n - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$ be its characteristic polynomial, then $p_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_0 = 0$.

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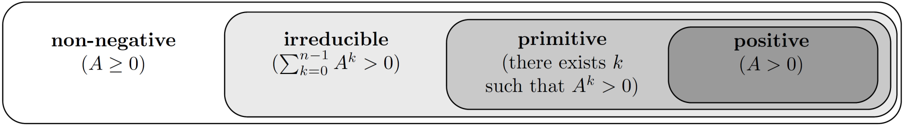

Perron-Frobenius theorem

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Let $A \in \mathbb{R}^{n \times n}$, $n \geq 2$. If A is nonnegative, then

1. there exists a real eigenvalue $\lambda \geq |\mu| \geq 0$ for all other eigenvalues μ
2. the right and left eigenvectors v, w of λ can be selected nonnegative

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If additionally A is irreducible, then

3. the eigenvalue λ is strictly positive and simple
4. the right and left eigenvectors v, w of λ are unique and positive

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- $\lambda = \rho(A)$ is called the dominant eigenvalue of A
- These are sufficient but not necessary conditions

Perron-Frobenius theorem: examples

- $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{spec}(A_1) = \{1, 1\}$

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- $A_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, $\text{spec}(A_4) = \{1, -\frac{1}{2}\}$

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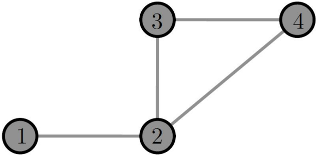
- $A_5 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\text{spec}(A_5) = \{2, 0\}$

Perron-Frobenius theorem: examples



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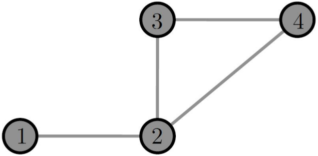
- Already know that 1 is an eigenvalue and spectral radius is 1
- Is it irreducible?



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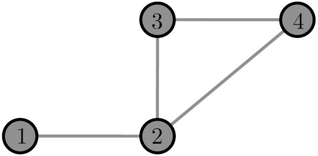
$$A^2 = \begin{bmatrix} 3/8 & 3/8 & 1/8 & 1/8 \\ 3/16 & 17/48 & 11/48 & 11/48 \\ 1/12 & 11/36 & 11/36 & 11/36 \\ 1/12 & 11/36 & 11/36 & 11/36 \end{bmatrix}$$

  - 1 is a simple eigenvalue (may exist other eigenvalues on the unit circle)



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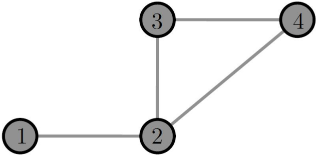

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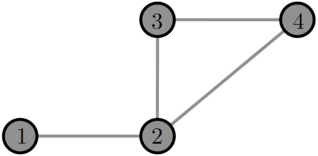
  - $1 > |\mu|$ for all other eigenvalues μ

Perron-Frobenius theorem: examples



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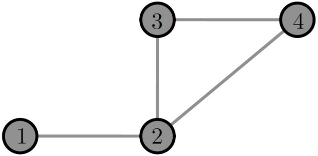
  - 1 is a simple eigenvalue (may exist other eigenvalues on the unit circle)
- Is it primitive?
  - $1 > |\mu|$ for all other eigenvalues μ
- It is semi-convergent

Powers of nonnegative matrices with a simple and dominant eigenvalue

Let $A \in \mathbb{R}^{n\times n}$, $n \geq 2$ be nonnegative with dominant eigenvalue λ and the right and left eigenvectors are denoted by v and w of λ, $v^\top w = 1$. If λ is simple and strictly larger in magnitude than all other eigenvalues, then we have

$$\lim_{k\to\infty} \frac{A^k}{\lambda^k} = vw^\top$$

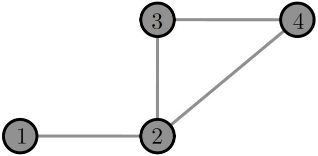
Proof: Jordan normal form



$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} = Ax(k)$$

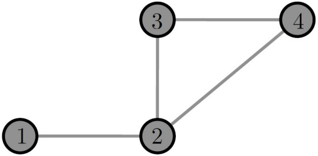
- The matrix is primitive
    - Dominant eigenvalue 1 is simple and strictly larger than others

Perron-Frobenius theorem: applications to matrix power



$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} = Ax(k)$$

- The matrix is primitive
  - Dominant eigenvalue 1 is simple and strictly larger than others
  - $A^k \to vw^\top$ where v and w are right and left eigenvectors and $v^\top w = 1$
- Let us verify $w = [1/6, 1/3, 1/4, 1/4]^\top$ is a left dominant eigenvector



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$$\lim_{k\to\infty} A^k = \mathbb{1}_4 w^\top = \begin{bmatrix} 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \end{bmatrix}$$

Average consensus cannot be reached!

Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation