# Elements of Graph Theory

## AU7036: Introduction to Multi-agent Systems

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February 23, 2024

# Last time

- Discrete-time linear systems
- Jordan normal form
- Row-stochastic matrices and their spectral radius
- Nonnegative matrices and Perron-Forbenius theorem

# Today

1. Graphs and digraphs
2. Walks and connectivity in undirected graphs
3. Walks and connectivity in digraphs
4. Weighted digraphs

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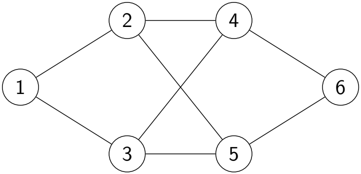
1. Graphs and digraphs
2. Walks and connectivity in undirected graphs
3. Walks and connectivity in digraphs
4. Weighted digraphs

- A (undirected) graph is a pair $G = (V, E)$ such that
  - $V$ is the set of nodes
  - $E$ is set of edges of form $\{u, v\}$ for $u, v \in V$ (unordered pair of nodes)

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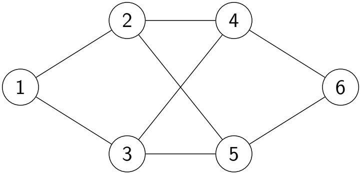


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  - e.g., nodes 1 and 2 are neighbors, nodes 1 and 5 are not neighbors

# Graphs, neighbors and degrees

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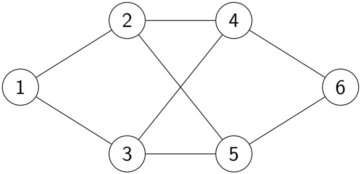


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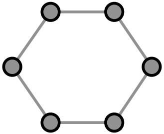
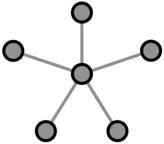
- Two nodes $u$ and $v$ are neighbors if $\{u, v\} \in E$
  - e.g., nodes 1 and 2 are neighbors, nodes 1 and 5 are not neighbors
- The degree of $v$ is the number of neighbors of $v$
  - e.g., degree of node 2 is 3, degree of node 6 is 2

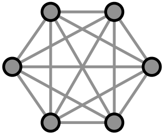
(a) Path graph



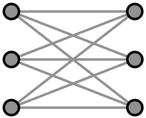
(b) Cycle graph



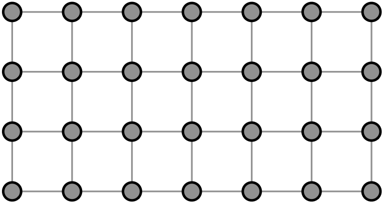
(c) Star graph



(d) Complete graph



(e) Complete bipartite graph



(f) Grid graph

# Digraphs, in/out-neighbors and in/out-degrees

- A directed graph (digraph) is a pair $G = (V, E)$ such that
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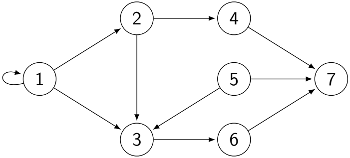


A directed graph with a *self-loop* at node 1

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$
$$E = \{(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 6), (4, 7), (5, 3), (5, 7), (6, 7)\}$$

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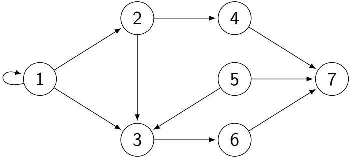


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- In $G$ with an edge $(u, v) \in E$
  - $u$ is an in-neighbor of $v$ (e.g., node 5 is an in-neighbor of node 7)
  - $v$ is an out-neighbor of $u$ (e.g., node 7 is an out-neighbor of node 4)

# Digraphs, in/out-neighbors and in/out-degrees

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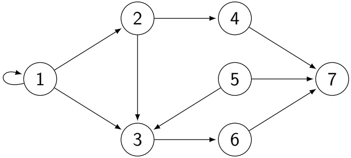


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  - $v$ is an out-neighbor of $u$ (e.g., node 7 is an out-neighbor of node 4)
- In-degree of $v$ is the number of in-neighbors of $v$ (e.g., $d_{\text{in}}(3) = 3$)
- Out-degree of $v$ is the number of out-neighbors (e.g., $d_{\text{out}}(3) = 1$)

# Subgraphs

- $G' = (V', E')$ is a subgraph of graph (digraph) $G = (V, E)$ if
  1. $V' \subset V$, and
  2. $E' \subset E$
- $G' = (V', E')$ is spanning subgraph of graph (digraph) $G = (V, E)$ if
  1. $V' = V$, and
  2. $E' \subset E$

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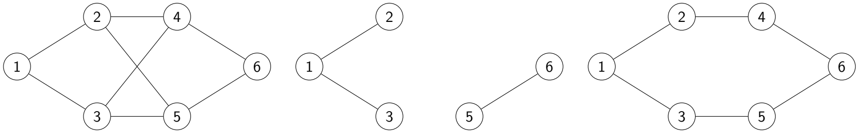


(a) Graph $G$ (b) A subgraph of $G$ (c) A spanning subgraph

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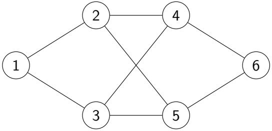
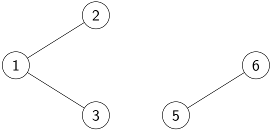
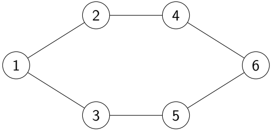


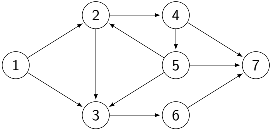

(a) Graph $G$



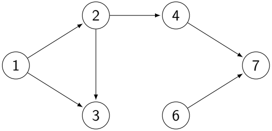

(b) A subgraph of $G$




(c) A spanning subgraph




(a) Directed graph $G$

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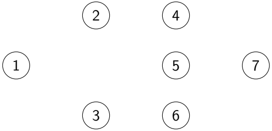
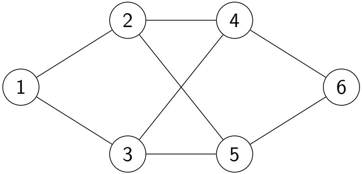



(c) A spanning subgraph

# Today

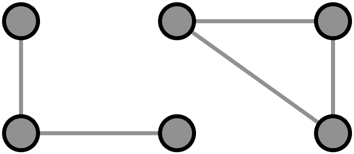
# Walks and cycles




- A walk (path) is sequence of nodes $(i_1, i_2, \ldots, i_r)$ such that
  - $i_k \in V$ for $k \in \{1, \ldots, r\}$
  - $\{i_k, i_{k+1}\} \in E$ for $k \in \{1, \ldots, r-1\}$

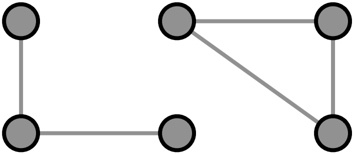
  e.g., $(1, 2)$, $(1, 2, 5, 6, 5, 3)$
- A simple walk is a walk with no node repetitions except first/last node
  e.g., $(1, 2)$, $(1, 2, 5, 6, 4)$, $(2, 5, 6, 4, 2)$
- A (simple) cycle is a simple walk starting and ending at same node
  e.g., $(1, 3, 5, 2, 1)$, $(2, 5, 6, 4, 2)$

# Connectivity and connected component



- A graph is connected if there exists a walk between any two nodes

# Connectivity and connected component



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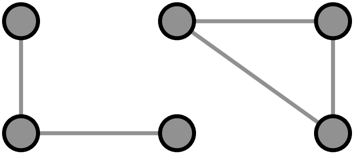
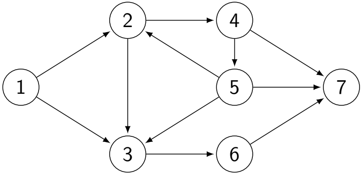


- A graph is connected if there exists a walk between any two nodes
- A connected component of a graph is a maximal connected subgraph
- A graph that contains no cycles is acyclic
- A connected acyclic graph is a tree

# Today

# Directed walks and cycles

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  e.g., $(1, 2)$, $(1, 2, 3, 6)$, $(2, 4, 5, 2, 3)$
- A simple directed walk is a directed walk with no node repetitions except first/last node
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- A (simple) cycle is a simple walk starting and ending at same node
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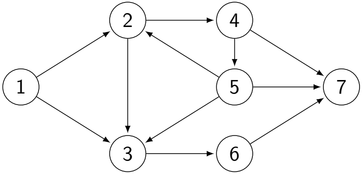


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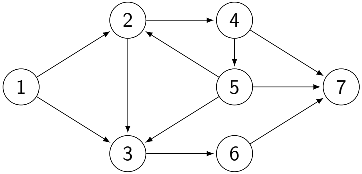
- A (simple) cycle is a simple walk starting and ending at same node
  e.g., $(2,4,5,2)$
- A digraph is acyclic if it contains no cycles

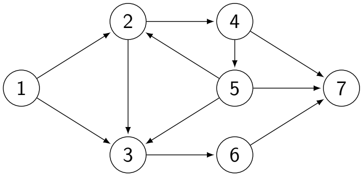
# Sources and sinks




- A source node is a node with in-degree 0
  e.g., node 1
- A sink node is a node with out-degree 0
  e.g., node 7

Are there graphs with no source and sink nodes?

# Sources and sinks

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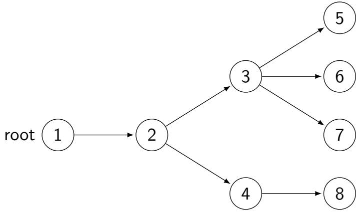
**Sources and sinks in directed acyclic graph (DAG)**

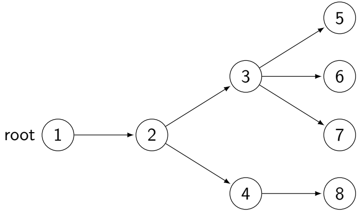
Every acyclic digraph has at least one source and at least one sink.

# Directed trees

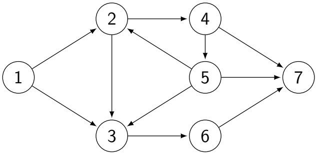



- A directed tree is an acyclic digraph with a special node called root such that every other node can be reached from root by a unique walk

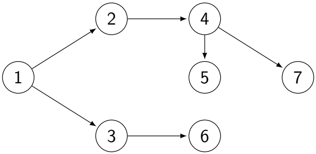
# Directed trees




- A directed tree is an acyclic digraph with a special node called root such that every other node can be reached from root by a unique walk
- A directed spanning tree of a graph is a spanning subgraph that is a directed tree

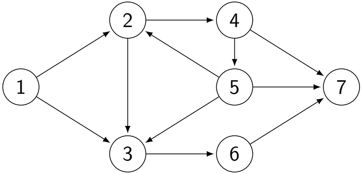



(a) Directed graph $G$



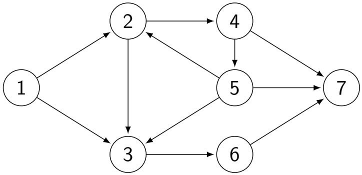

(b) A directed spanning tree

# Connectivity




Four notions of connectivity

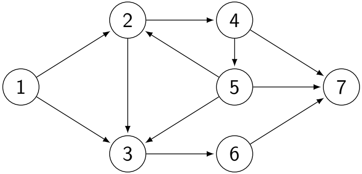
1. A digraph is strongly connected if walk exists between any two nodes




Four notions of connectivity

1. A digraph is strongly connected if walk exists between any two nodes
2. A digraph is weakly connected if undirected version is connected

# Connectivity




Four notions of connectivity

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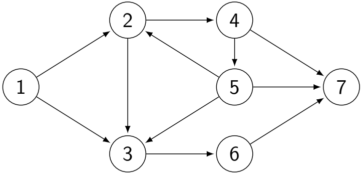
3. A digraph possesses a globally reachable node if one node can be reached from all other nodes

# Connectivity




Four notions of connectivity

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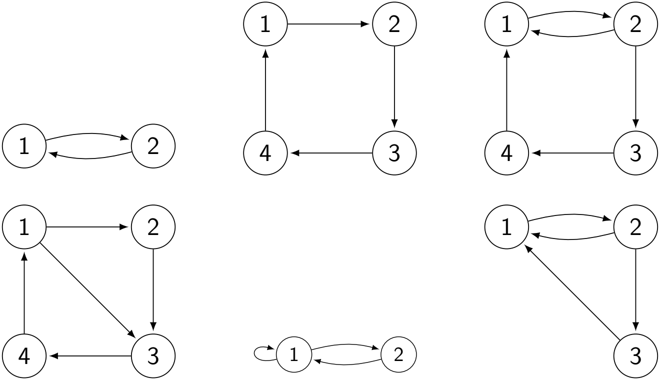
3. A digraph possesses a globally reachable node if one node can be reached from all other nodes
4. A digraph possesses a directed spanning tree if one node can reach all other nodes

# Periodicity of strongly connected digraphs

- The period of a strongly connected digraph is the greatest common division of the lengths of all simple cycles



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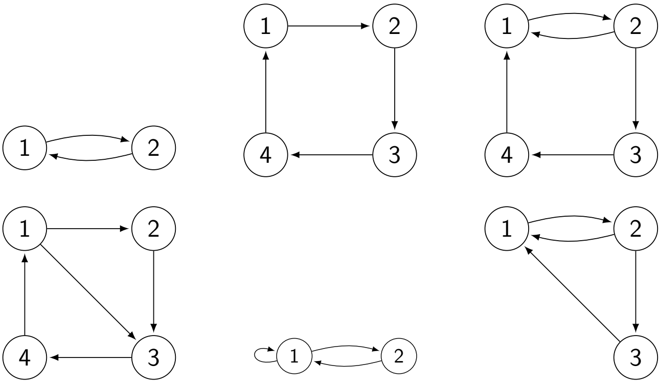
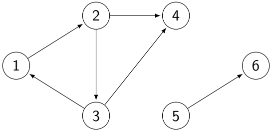


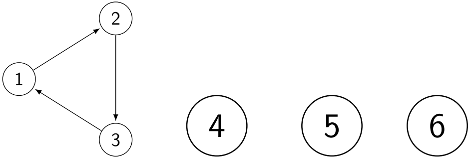
- A digraph is periodic if its period is $> 1$; otherwise, it is aperiodic

Other equivalent definitions?

- A strongly connected component of a digraph is a maximal strongly connected subgraph

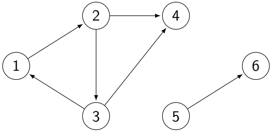



(a) Digraph $G$



# Condensation digraphs

- A strongly connected component of a digraph is a maximal strongly connected subgraph

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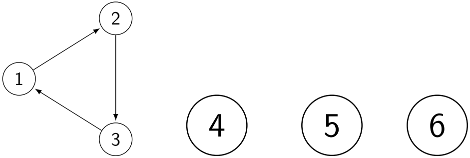
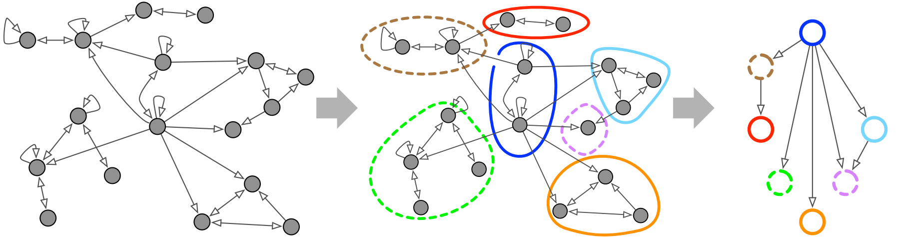



- The condensation digraph $C(G) = (\mathcal{H}, R)$ of digraph $G = (V, E)$:
  - $\mathcal{H}$ is the set of strongly connected components of $G$
  - An edge $(H_1, H_2) \in R$ exists if $(u, v) \in E$ for some $u \in H_1$ and $v \in H_2$
  - $G$ does not have self-loops

# Condensation digraphs

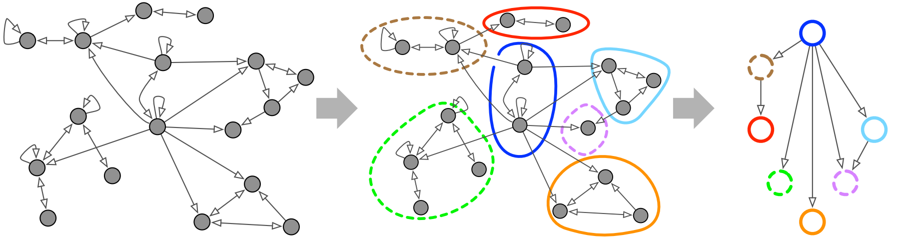


(a) A digraph $G$

(b) strongly connected components of $G$

(c) Condensation digraph $C(G)$

# Condensation digraphs



(a) A digraph $G$

(b) strongly connected components of $G$

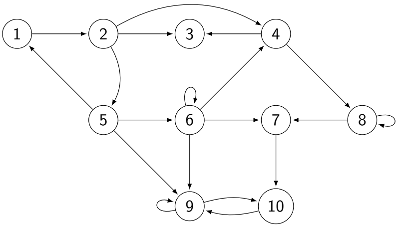
(c) Condensation digraph $C(G)$

## Properties of condensation digraphs

The condensation digraph $C(G)$ is acyclic, and following are equivalent

- $G$ contains a globally reachable node
- $C(G)$ contains a globally reachable node
- $C(G)$ contains a unique sink

# Condensation digraphs: exercise 1



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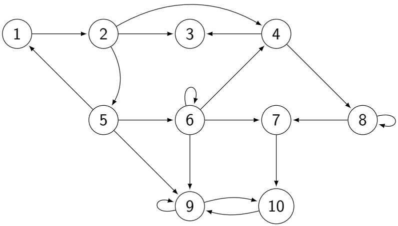


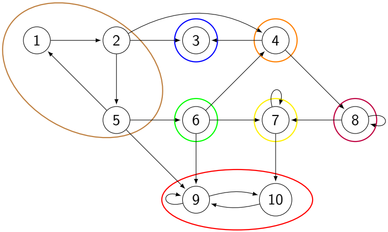

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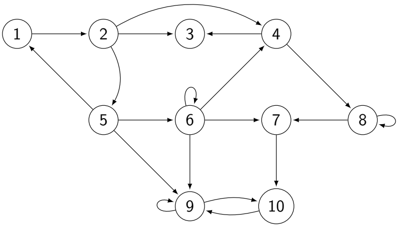
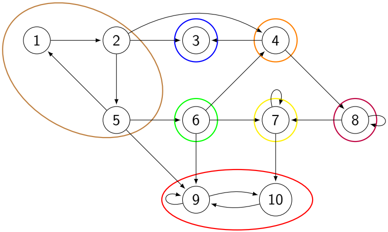
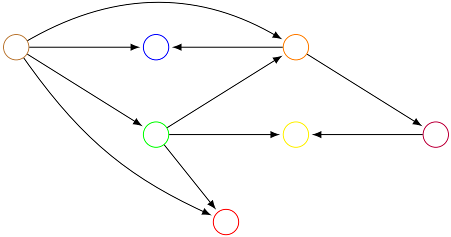
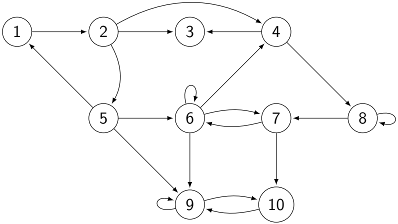


# Condensation digraphs: exercise 2



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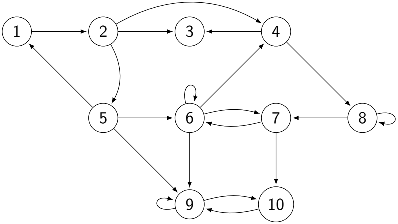


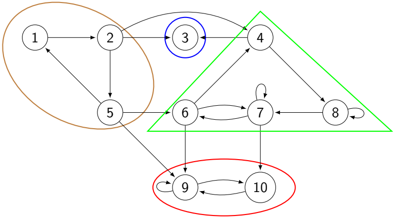

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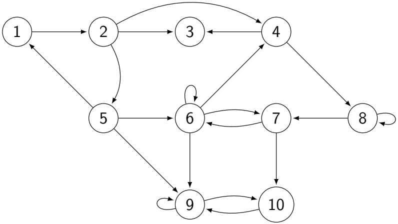
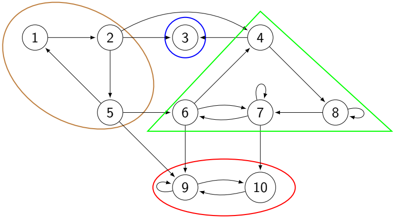
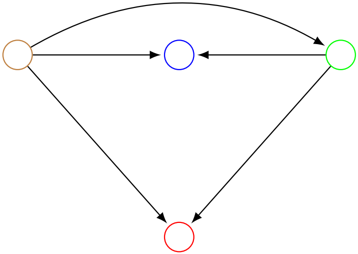


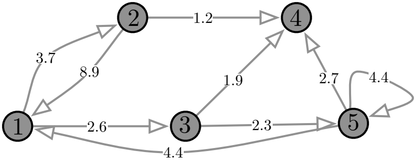



# Today

# Weighted digraphs

- A weighted digraph is a triplet $G = (V, E, \{a_e\}_{e \in E})$ such that
  - $V$ is the set of nodes
  - $E \subset V \times V$ is a set of edges (ordered pair of nodes)
  - $\{a_e\}_{e \in E}$ is a collection of strictly positive weights for the edges $E$




$$a_{12} = 3.7,\ a_{13} = 2.6,\ a_{21} = 8.9,$$
$$a_{24} = 1.2,\ a_{34} = 1.9,\ a_{35} = 2.3,$$
$$a_{51} = 4.4,\ a_{54} = 2.7,\ a_{55} = 4.4.$$

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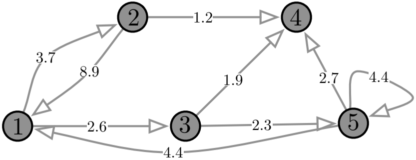


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- Weighted in-degree and Weighted out-degree of $v_i$ are defined by

$$d_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ij} \qquad d_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ji}$$

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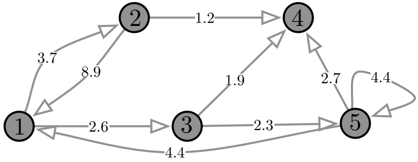


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- A weighted digraph is weight-balanced if for $v_i \in V$, $d_{\text{in}}(v_i) = d_{\text{out}}(v_i)$

# Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation