Elements of Algebraic Graph Theory

AU7036: Introduction to Multi-agent Systems

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Last time

- Graphs and directed graphs
  - Definitions, neighbors, degrees, subgraphs
- Walks and connectivity in undirected graphs
  - Walks, cycles, connected components, acyclicity, trees
- Walks and connectivity in digraphs
  - Directed walks and cycles, sources/sinks, DAG, directed tree
  - Strong/weak connectivity, spanning tree, globally reachable node
  - Period, strongly connected components
  - Condensation digraph
- Weighted digraphs

Today

1. The adjacency matrix
2. Algebraic graph theory: graphs and adjacency matrices
3. Graph theoretical characterization of special matrices
4. Elements of spectral graph theory

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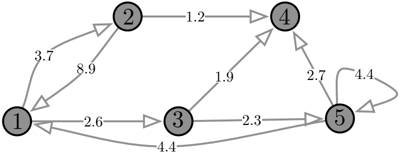
Weighted digraph and adjacency matrix

- Given a weighted digraph $G = (V, E, \{a_e\}_{e \in E})$, the weighted adjacency matrix A satisfies
  - $A_{ij} = a_{(i,j)}$ if $(i,j) \in E$
  - $A_{ij} = 0$ otherwise



$$A = \begin{bmatrix} 0 & 3.7 & 2.6 & 0 & 0 \\ 8.9 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.9 & 2.3 \\ 0 & 0 & 0 & 0 & 0 \\ 4.4 & 0 & 0 & 2.7 & 4.4 \end{bmatrix}$$

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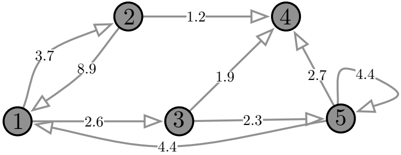

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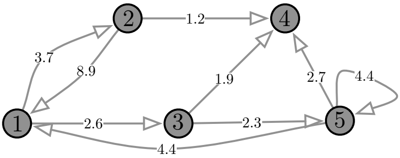
- Given a weighted digraph $G = (V, E, \{a_e\}_{e \in E})$, the binary adjacency matrix $A \in \{0,1\}^{n \times n}$ satisfies

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Weighted degrees



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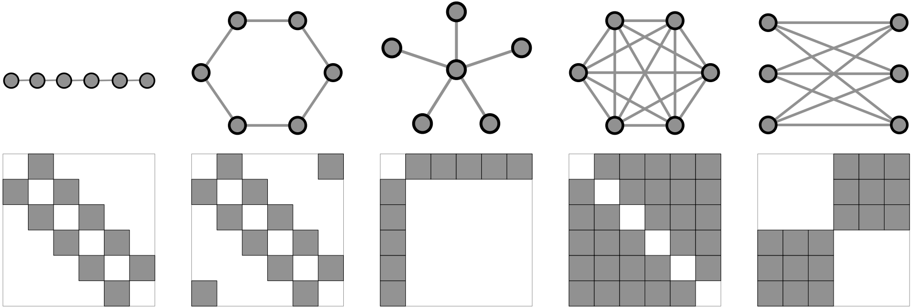
- Given a weighted digraph G and adjacency matrix A
  - Weighted out-degree matrix

$$D_{\text{out}} = \text{diag}(A\mathbb{1}_n) = \begin{bmatrix} d_{\text{out}}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{\text{out}}(n) \end{bmatrix}$$

  - Weighted in-degree matrix

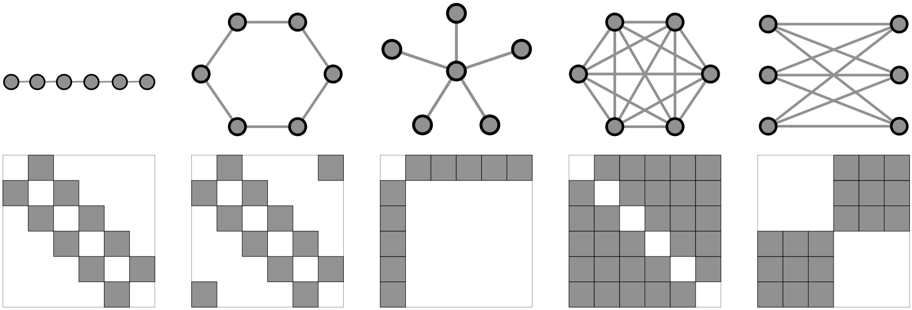
$$D_{\text{in}} = \text{diag}(A^\top \mathbb{1}_n) = \begin{bmatrix} d_{\text{in}}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{\text{in}}(n) \end{bmatrix}$$

Adjacency matrix: examples



(a) P_6 (b) C_6 (c) S_6 (d) K_6 (e) $K_{3,3}$

- These are undirected graphs



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- These are undirected graphs
- Why do the patterns look so "pretty"?

Useful concepts: permutation matrices

- A permutation matrix is a square binary matrix with precisely one 1 in every row and column

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

  - $P^{-1} = P^{\top}$
  - $P^{\top}AP$ is a similarity transformation and reorder rows and columns of A

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad P^{\top}AP = \begin{bmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{bmatrix}$$

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Basic results

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- Each node has weighted out-degree 1 $\iff A$ is row-stochastic

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- Each node has weighted out-degree 1 $\iff$ A is row-stochastic
- Each node has weighted out- and in-degree 1 $\iff$ A is doubly-stochastic

Matrix powers and directed walks

Let A be an adjacency matrix

- We first note (obviously) $A_{ij} > 0$ if and only if
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- there exists a node h such that $(A)_{ih} > 0$ and $(A)_{hj} > 0$
- (i, h) and (h, j) are edges of G
- there exists a directed walk of length 2 from i to j

Matrix powers and directed walks

Directed walks and powers of the adjacency matrix

Let G be a weighted digraph with n nodes, with adjacency matrix A and binary adjacency matrix $A_{0,1} \in \{0, 1\}^{n \times n}$. For all $i, j \in \{1, \ldots, n\}$ and $k \geq 1$

1. $(A_{0,1}^k)_{ij}$ equals number of walks of length k from i to j;
2. $(A^k)_{ij} > 0$ if and only if there exists a walk of length k from i to j.

Today

Useful concepts: block triangular matrices and partitions

- An $n \times n$ matrix A is block triangular if there exists r

$$A = \left[\begin{array}{c|c} B_{r\times r} & C_{r\times(n-r)} \\ \hline \mathbb{0}_{(n-r)\times r} & D_{(n-r)\times(n-r)} \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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- $\{I, J\}$ is a partition of the index set $\{1, \ldots, n\}$ if
  1. $I \cup J = \{1, \ldots, n\}$
  2. $I \neq \emptyset$, $J \neq \emptyset$,
  3. $I \cap J = \emptyset$

Characterization of irreducible matrices

Strongly connected digraphs and irreducible adjacency matrices

Let G be a weighted digraph with $n \geq 2$ nodes and with weighted adjacency matrix A. The following are equivalent:

1. A is irreducible, that is, $\sum_{k=0}^{n-1} A^k > 0$;
2. G is strongly connected;

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3. there exists no permutation matrix P such that $PAP^\top$ is block triangular (alternative definition);
4. for all partitions $\{I, J\}$ of the index set $\{1, \ldots, n\}$, there exists $i \in I$ and $j \in J$ such that (i, j) is a directed edge in G.

Global reachability and adjacency matrix

Global reachability and powers of the adjacency matrix

Let G be a weighted digraph with $n \geq 2$ nodes and with weighted adjacency matrix A. For any $j \in \{1, \ldots. n\}$, the following are equivalent:

1. the jth node is globally reachable,
2. the jth column of $\sum_{k=0}^{n-1} A^k$ is positive.

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Connectivity and positive powers of the adjacency matrix

Let G be a weighted digraph with $n \geq 2$ nodes, weighted adjacency matrix A, and a self-loop at each node. The following are equivalent:

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2. A^{n-1} is positive, that is, A is primitive.

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Strongly connected aperiodic digraphs and primitive adjacency matrices

Let G be a weighted digraph with $n \geq 2$ nodes and with weighted adjacency matrix A. The following are equivalent:

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Proof sketch

Frobenius number

Given a finite set $A = \{a_1, a_2, \ldots, a_n\}$ of positive integers, an integer M is said to be representable by A if there exist non-negative integers $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $M = \alpha_1 a_1 + \cdots + \alpha_n a_n$. The following statements are equivalent:

1. there exists a (finite) largest unrepresentable integer, called the Frobenius number of A,
2. the greatest common divisor of A is 1 (coprime).

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Bounds on spectral radius I

Bounds on the spectral radius of non-negative matrices, I

For a non-negative matrix $A \in \mathbb{R}^{n \times n}_{\geq 0}$, vector $x \in \mathbb{R}^{n}_{\geq 0}$, $x \neq \mathbb{0}_n$, and scalars $r_1, r_2 > 0$, the following statements hold:

1. if $r_1 x \leq Ax$, then $r_1 \leq \rho(A)$,
2. if $Ax \leq r_2 x$ and $x \in \mathbb{R}^{n}_{>0}$, then $\rho(A) \leq r_2$.

Moreover, for an irreducible matrix A,

1. if $r_1 x \leq Ax \leq r_2 x$, $r_1 x \neq Ax \neq r_2 x$, then $r_1 < \rho(A) < r_2$ and $x > 0$.

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Monotonicity of spectral radius of non-negative matrices

Let A and A' be non-negative matrices in $\mathbb{R}^{n\times n}_{\geq 0}$. Then,

1. if $A \leq A'$, then $\rho(A) \leq \rho(A')$,
2. if additionally $A \neq A'$ and A' is irreducible, then $\rho(A) < \rho(A')$.

Bounds on spectral radius II

Bounds on the spectral radius of non-negative matrices, II

For a non-negative matrix $A \in \mathbb{R}^{n \times n}_{\geq 0}$ with associated digraph G, the following statements hold:

1. $\min(A\mathbb{1}_n) \leq \rho(A) \leq \max(A\mathbb{1}_n)$; and
2. if $\min(A\mathbb{1}_n) < \max(A\mathbb{1}_n)$ then the following statements are equivalent:
   1. for each node i with $\mathbb{e}_i^\top A\mathbb{1}_n = \max(A\mathbb{1}_n)$, there exists a directed walk in G from node i to a node j with $\mathbb{e}_j^\top A\mathbb{1}_n < \max(A\mathbb{1}_n)$,
   2. $\rho(A) < \max(A\mathbb{1}_n)$.

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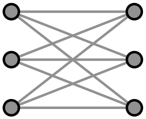
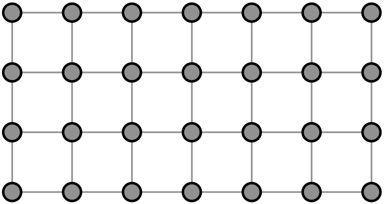
   2. $\rho(A) < \max(A\mathbb{1}_n)$.



(a) Complete bipartite graph



(b) Grid graph

Row-substochastic matrix and their spectral radius

Row-substochastic matrix

A non-negative matrix $A \in \mathbb{R}^{n \times n}$ is row-substochastic if its row-sums are at most 1 and at least one row-sum is strictly less than 1, that is,

$$A\mathbb{1}_n \leq \mathbb{1}_n, \text{ and there exists } i \in \{1, \dots, n\} \text{ such that } \mathbb{e}_i^\top A\mathbb{1}_n < 1.$$

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Convergent row-substochastic matrices

Let A be row-substochastic with associated digraph G.

1. A is convergent if and only if G contains directed walks from each node with out-degree 1 to a node with out-degree less than 1,
2. if A is irreducible, then A is convergent.

Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation