Discrete-time Averaging Systems

AU7036: Introduction to Multi-agent Systems

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Last time

- The adjacency matrix
- Graph and adjacency matrix
  - Matrix power and directed walks in graphs
- Graph theoretical characterization of matrices
  - Strong connectivity $\iff$ irreducibility
  - Strong connectivity + aperiodicity $\iff$ primitivity
- Spectral graph theory
  - Monotonicity of spectral radius
  - Convergence of substochastic matrices

Today

1. Averaging systems achieving asymptotic consensus
2. Averaging systems achieving asymptotic disagreement
3. Consensus via disagreement and Lyapunov functions
4. Design of graph weights

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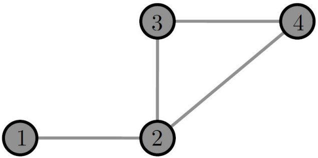
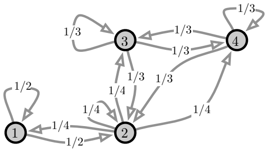
Asymptotic consensus: example 1



$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$



- We know that

$$\lim_{k\to\infty} A^k = \mathbb{1}_n w^\top$$

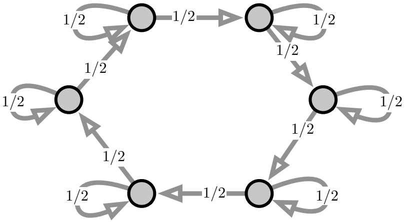
where $w = [1/6, 1/3, 1/4, 1/4]^\top$ is the left dominant eigenvector of A

- State achieves consensus:

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) = (\mathbb{1}_4 w^\top)x(0) = (w^\top x(0))\mathbb{1}_4 = \begin{bmatrix} w^\top x(0) \\ \vdots \\ w^\top x(0) \end{bmatrix}$$

Average consensus is not achieved.

Asymptotic consensus: example 2

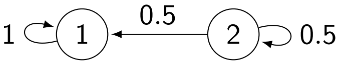


$$A = \begin{bmatrix} 1/2 & 1/2 & \cdots & 0 & 0 \\ 0 & 1/2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1/2 & 1/2 \\ 1/2 & 0 & \cdots & 0 & 1/2 \end{bmatrix}$$

- A is primitive
- A is doubly stochastic, i.e., its left dominant eigenvector $w = \frac{1}{6}\mathbb{1}_n$
- State achieves average consensus

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) = \text{average}\big(x(0)\big)\mathbb{1}_n$$

Asymptotic consensus: example 3



- Associated averaging system

$$x_1(k+1) = x_1(k)$$
$$x_2(k+1) = 0.5x_1(x) + 0.5x_2(k)$$

or in matrix form

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix} x(k)$$

- State achieves consensus

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} A^k x(0) = x_1(0)\mathbb{1}_n$$

Primitivity (even strong connectivity) is not necessary for consensus.

Consensus for discrete-time averaging systems

Equivalent characterizations for consensus

Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:

(A1) the eigenvalue 1 is simple and all other eigenvalues μ satisfy $|\mu| < 1$;

Consensus for discrete-time averaging systems

Equivalent characterizations for consensus

Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:

(A1) the eigenvalue 1 is simple and all other eigenvalues μ satisfy $|\mu| < 1$;

(A2) A is semi-convergent and

$$\lim_{k\to\infty} A^k = \mathbb{1}_n w^\top$$

where $w \geq 0$, $w^\top A = w^\top$ and $w^\top \mathbb{1}_n = 1$;

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(A3) G contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.

A is called indecomposable if any of the above holds

Consensus for discrete-time averaging systems

Row-stochastic matrices with a globally-reachable aperiodic SCC

Let A be a row-stochastic matrix and let G be its associated digraph. If any of (A1)-(A3) holds (or A is indecomposable), then

1. left dominant eigenvector $w \geq 0$, $w_i > 0$ iff i is globally reachable;

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1. left dominant eigenvector $w \geq 0$, $w_i > 0$ iff i is globally reachable;
2. the solution to the averaging model $x(k+1) = Ax(k)$ satisfies

$$\lim_{k \to \infty} x(k) = \left(w^\top x(0)\right) \mathbb{1}_n;$$

Consensus for discrete-time averaging systems

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$$\lim_{k \to \infty} x(k) = \left(w^\top x(0)\right) \mathbb{1}_n;$$

3. if additionally A is doubly-stochastic, then $w = \frac{1}{n}\mathbb{1}_n$ and

$$\lim_{k \to \infty} x(k) = \frac{\mathbb{1}_n^\top x(0)}{n} \mathbb{1}_n = \text{average}\big(x(0)\big)\mathbb{1}_n.$$

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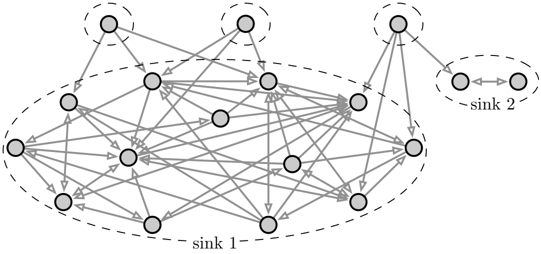
Graph condensation with multiple sinks



- There are two sinks in the condensation graph
- No "information exchange" between these sinks

Equivalent characterizations for disagreement

Let A be a row-stochastic matrix, G be its associated digraph, and $n_s \geq 2$ be the number of sinks in the condensation digraph $C(G)$. The following statements are equivalent:

(A1) the eigenvalue 1 is semisimple with multiplicity n_s and all other eigenvalues μ satisfy $|\mu| < 1$;

Disagreement for discrete-time averaging systems

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(A2) A is semi-convergent;

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(A1) the eigenvalue 1 is semisimple with multiplicity n_s and all other eigenvalues μ satisfy $|\mu| < 1$;

(A2) A is semi-convergent;

(A3) each sink of $C(G)$, regarded as a subgraph of G, is aperiodic.

Disagreement for discrete-time averaging systems

Row-stochastic matrices with multiple aperiodic sinks

Let A be a row-stochastic matrix, G be its associated digraph, and $n_s \geq 2$ be the number of sinks in the condensation digraph $C(G)$. If any of (A1)-(A3) holds, then

1. left dominant eigenvectors $w^p \in \mathbb{R}^n$, $p \in \{1, \dots, n_s\}$ of A can be selected to satisfy: $w^p \geq 0$, $\mathbb{1}_n^\top w^p = 1$ and $w_i^p > 0$ iff node i belongs to sink p;

Disagreement for discrete-time averaging systems

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2. the solution to the averaging model $x(k+1) = Ax(k)$ satisfies

$$\lim_{k\to\infty} x_i(k) = \begin{cases} (w^p)^\top x(0), & \text{if node } i \text{ belongs to sink } p, \\ \displaystyle\sum_{p=1}^{n_s} z_{i,p}\big((w^p)^\top x(0)\big), & \text{otherwise} \end{cases}$$

where $z_{i,p}$, $p \in \{1, \dots, n_s\}$, are convex combination coefficients and $z_{i,p} > 0$ iff there exists a directed walk from node i to the sink p.

Today

Disagreement and deflated matrices

For the averaging system

$$x(k+1) = Ax(k)$$

- "Suppose" we know that $x(k) \xrightarrow{k\to\infty} (w^\top x(0))\mathbb{1}_n$
- We can define the disagreement vector $\delta(k)$ as follows

$$\delta(k) = x(k) - (w^\top x(0))\mathbb{1}_n$$

- The disagreement vector satisfies

$$\delta(k+1) = (A - \mathbb{1}w^\top)\delta(k)$$

where $A - \mathbb{1}w^\top$ is called deflated matrix

Convergence of disagreement vector

$$\delta(k+1) = (A - \mathbb{1}w^\top)\delta(k)$$

Convergence of disagreement vector

Given row-stochastic matrix A with left dominant eigenvector w, $\mathbb{1}^\top w = 1$,

- if A is primitive, then the deflated matrix $A - \mathbb{1}w^\top$ has the same eigenvalues and eigenvectors of A, except the eigenvalue 1 which is replaced by 0 (with same right and left eigenvectors);

Convergence of disagreement vector

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Convergence of disagreement vector

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- if A is primitive, then the deflated matrix $A - \mathbb{1}w^\top$ has the same eigenvalues and eigenvectors of A, except the eigenvalue 1 which is replaced by 0 (with same right and left eigenvectors);
- if A is primitive, then
$$\rho(A - \mathbb{1}w^\top) < 1$$

The disagreement vector may not vanish monotonically.

Quadratic disagreement

- Define the quadratic disagreement function $V_{\text{qd}} : \mathbb{R}^n \to \mathbb{R}$ by

$$V_{\text{qd}}(x) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (x_i - x_j)^2$$

where $V_{\text{qd}}(x) \geq 0$ and $V_{\text{qd}}(x) = 0$ iff x is a consensus vector

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- $V_{\text{qd}}(x)$ can be written in standard quadratic form as

$$V_{\text{qd}}(x) = x^\top (I_n - \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top) x$$

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- For symmetric row-stochastic matrix $A = A^\top$, define 2-coefficient of ergodicity by

$$\tau_2(A) = \|A - \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top\|_2 = \max_{\|y\|_2 = 1, y \perp \mathbb{1}_n} \|Ay\|_2$$

Convergence of quadratic disagreement

Given symmetric row-stochastic matrix $A = A^\top$ with associated graph G,

1. for all $x \in \mathbb{R}^n$
$$V_{\mathrm{qd}}(Ax) \leq (\tau_2(A))^2 V_{\mathrm{qd}}(x)$$
2. if G is connected, $\tau_2(A) < 1$;

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2. if G is connected, $\tau_2(A) < 1$;
3. if G is connected, then any solution to $x(k+1) = Ax(k)$ satisfies
$$V_{\mathsf{qd}}(x(k)) \le \underbrace{\tau_2(A)}_{<1}{}^{2k} V_{\mathsf{qd}}(x(0)), \quad \text{for all } k \in \mathbb{N}.$$

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The quadratic disagreement diminishes monotonically (Lyapunov function).

Max-min disagreement

- Define the max-min disagreement function $V_{\text{max-min}} : \mathbb{R}^n \to \mathbb{R}$ by

$$V_{\text{max-min}}(x) = \max_{i\in\{1,\dots,n\}} x_i - \min_{i\in\{1,\dots,n\}} x_i = \max_{i,j\in\{1,\dots,n\}} (x_i - x_j)$$

where $V_{\text{max-min}}(x) \geq 0$ and $V_{\text{max-min}}(x) = 0$ iff x is a consensus vector

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- For row-stochastic matrix A, define 1-coefficient of ergodicity by

$$\begin{aligned}
\tau_1(A) &= \max_{\|y\|_1 = 1, y \perp \mathbb{1}_n} \|A^\top y\|_1 \\
&= \frac{1}{2} \max_{i,j \in \{1,\dots,n\}} \sum_{h=1}^{n} |a_{ih} - a_{jh}| \\
&= 1 - \min_{i,j \in \{1,\dots,n\}} \sum_{h=1}^{n} \min\{a_{ih}, a_{jh}\}.
\end{aligned}$$

Convergence of max-min disagreement

Given a row-stochastic matrix A with associated graph G,

1. for all $x \in \mathbb{R}^n$
$$V_{\text{max-min}}(Ax) \leq \tau_2(A) V_{\text{max-min}}(x)$$
2. $\tau_1(A) < 1$ iff A is scrambling, i.e., any two nodes have a common out-neighbor;

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2. $\tau_1(A) < 1$ iff A is scrambling, i.e., any two nodes have a common out-neighbor;
3. if G contains a globally reachable node in h steps, A^h is scrambling
$$V_{\text{max-min}}(x(k)) \leq (\tau_1(A^h))^{\lfloor k/h \rfloor} V_{\text{max-min}}(x(0)), \quad \text{for all } k \in \mathbb{N}.$$

Convergence of max-min disagreement

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The max-min disagreement diminishes monotonically (Lyapunov function).

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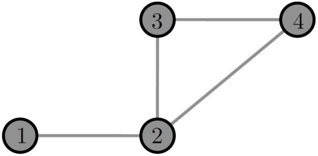
The equal-neighbor model



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

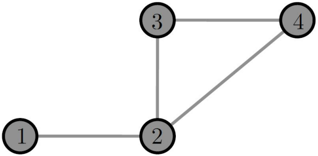
- Let G be a connected undirected graph, the equal-neighbor model:

$$A_{\text{equal-nghbr}} = D^{-1}A$$

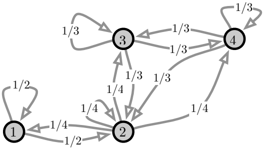
$D = \text{diag}(d_1, \dots, d_n)$ and A are degree and 0-1 adjacency matrices

- The left dominant eigenvector is

$$w_{\text{equal-nghbr}} = \frac{1}{\sum_{i=1}^{n} d_i} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$



$$A_{\text{wsn}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$



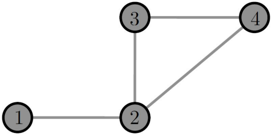
- The update matrix can be written as

$$A_{\text{wsn}} = (D + I_4)^{-1}(A + I_4)$$

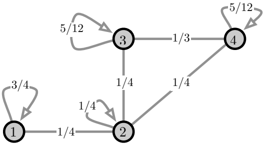
- The left dominant eigenvector is

$$w_{\text{wsn}} = \frac{1}{n + \sum_{i=1}^{n} d_i} \begin{bmatrix} d_1 + 1 \\ \vdots \\ d_n + 1 \end{bmatrix}$$

The Metropolis–Hastings model

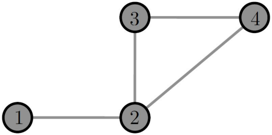


$$A_{\mathrm{MH}} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{5}{12} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \end{bmatrix}$$



- Let G be undirected graph, the Metropolis–Hastings model:

$$(A_{\mathrm{MH}})_{ij} = \begin{cases} \dfrac{1}{1 + \max\{d_i, d_j\}}, & \text{if } \{i, j\} \in E \text{ and } i \neq j, \\ 1 - \displaystyle\sum_{\{i,h\} \in E, h \neq i} (A_{\mathrm{MH}})_{ih}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$



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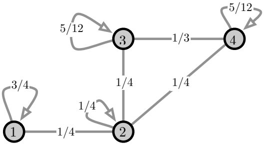

- Let G be undirected graph, the Metropolis–Hastings model:

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Properties of A_{MH}

1. $A_{\mathrm{MH}} = A_{\mathrm{MH}}^\top$
2. A_{MH} is primitive iff G is connected

Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation