The Laplacian Matrix

AU7036: Introduction to Multi-agent Systems

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Last time

- Averaging systems achieving asymptotic consensus
  1. Eigenvalue 1 is simple, all others strictly less than 1
  2. $\lim_{k\to\infty} A^k = \mathbb{1}_n w^\top$
  3. G contains a globally reachable node and the subgraph is aperiodic
- Averaging systems achieving asymptotic disagreement
  1. Eigenvalue 1 is semisimple, all others strictly less than 1
  2. A is semiconvergent
  3. Each sink is aperiodic
- Consensus via Lyapunov function
  - Disagreement vector
  - Quadratic disagreement function
  - Max-min disagreement function
- Weight design
  - Equal-neighbor model
  - Metropolis-Hastings

Today

1. Definition, useful equalities and applications
2. Properties of Laplacian matrices
3. Symmetric Laplacian and algebraic connectivity
4. Laplacian systems and Laplacian pseudoinverses

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2. Properties of Laplacian matrices
3. Symmetric Laplacian and algebraic connectivity
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The Laplacian matrix: definition

Laplacian matrices

Given a weighted digraph G with adjacency matrix A and out-degree matrix $D_{\text{out}} = \text{diag}(A\mathbb{1}_n)$, the Laplacian matrix of G is

$$L = D_{\text{out}} - A.$$

In components $L = (\ell_{ij})_{i,j\in\{1,\dots,n\}}$

$$\ell_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j, \\ \displaystyle\sum_{h=1,h\neq i}^{n} a_{ih}, & \text{if } i = j. \end{cases}$$

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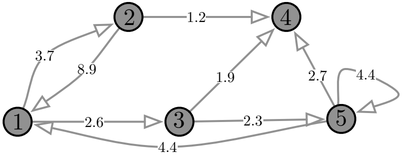
- Diagonal elements are nonnegative, off-diagonal elements nonpositive
- L does not depend on self-loops
- G is undirected (A symmetric) iff L is symmetric
- L is irreducible if G is strongly connected



$$L = \begin{bmatrix} 6.3 & -3.7 & -2.6 & 0 & 0 \\ -8.9 & 10.1 & 0 & -1.2 & 0 \\ 0 & 0 & 4.2 & -1.9 & -2.3 \\ 0 & 0 & 0 & 0 & 0 \\ -4.4 & 0 & 0 & -2.7 & 7.1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 3.7 & 2.6 & 0 & 0 \\ 8.9 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 1.9 & 2.3 \\ 0 & 0 & 0 & 0 & 0 \\ 4.4 & 0 & 0 & 2.7 & 4.4 \end{bmatrix} \qquad D_{\text{out}} = \begin{bmatrix} 6.3 & 0 & 0 & 0 & 0 \\ 0 & 10.1 & 0 & 0 & 0 \\ 0 & 0 & 4.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11.5 \end{bmatrix}$$

The Laplacian matrix: useful equalities

$$
\begin{aligned}
(Lx)_i &= \sum_{j=1}^{n} \ell_{ij} x_j \\
&= \ell_{ii} x_i + \sum_{j=1, j\neq i}^{n} \ell_{ij} x_j \\
&= \Big(\sum_{j=1, j\neq i}^{n} a_{ij} \Big) x_i + \sum_{j=1, j\neq i}^{n} (-a_{ij}) x_j \\
&= \sum_{j=1, j\neq i}^{n} a_{ij} (x_i - x_j) \\
&= \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij} (x_i - x_j)
\end{aligned}
$$

$(Lx)_i$ is the weighted sum of pairwise differences between i and neighbors

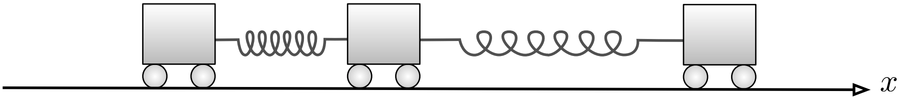
The Laplacian matrix: useful equalities

Suppose L is symmetric

$$
\begin{aligned}
x^\top L x &= \sum_{i=1}^n x_i (Lx)_i = \sum_{i=1}^n x_i \Big(\sum_{j=1, j\neq i}^n a_{ij}(x_i - x_j) \Big) \\
&= \sum_{i,j=1}^n a_{ij} x_i (x_i - x_j) = \Big(\frac{1}{2} + \frac{1}{2}\Big) \sum_{i,j=1}^n a_{ij} x_i^2 - \sum_{i,j=1}^n a_{ij} x_i x_j \\
&\overset{\text{by symmetry}}{=} \frac{1}{2} \sum_{i,j=1}^n a_{ij} x_i^2 + \frac{1}{2} \sum_{i,j=1}^n a_{ij} x_j^2 - \sum_{i,j=1}^n a_{ij} x_i x_j \\
&= \frac{1}{2} \sum_{i,j=1}^n a_{ij} (x_i - x_j)^2 = \sum_{\{i,j\} \in E} a_{ij} (x_i - x_j)^2.
\end{aligned}
$$

The function $x \mapsto x^\top L x$ is called the Laplacian potential function

The Laplacian in mechanical networks of springs



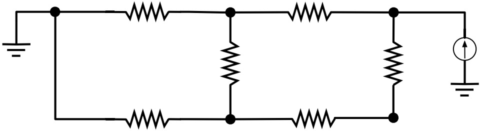
- $x_i \in \mathbb{R}$ denote the displacement of the ith rigid body.
- Ideal spring with spring constant a_{ij} connects the ith and jth bodies
- Each node (body) is subject to a force

$$F_i = \sum_{j \neq i} a_{ij}(x_j - x_i) = -(L_{\text{stiffness}} x)_i$$

- The potential energy is given by

$$E_{\text{elastic}} = \frac{1}{2} \sum_{\{i,j\} \in E} a_{ij}(x_i - x_j)^2 = \frac{1}{2} x^\top L_{\text{stiffness}} x$$

The Laplacian in electrical networks of resistors



- Each edge is a resistor with resistance r_{ij} between nodes i and j
- Ohm's law along each edge $\{i, j\}$ gives the current flowing from i to j

$$c_{i \to j} = \frac{v_i - v_j}{r_{ij}} \triangleq a_{ij}(v_i - v_j)$$

- Kirchhoff's current law at each node i:

$$c_{\text{injected at } i} = \sum_{j=1, j \neq i}^{n} a_{ij}(v_i - v_j) \implies c_{\text{injected}} = L_{\text{conductance}} v$$

- Energy dissipation

$$E_{\text{dissipated}} = \sum_{\{i,j\} \in E} a_{ij}(v_i - v_j)^2 = v^\top L_{\text{conductance}} v$$

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Row and column sums

Zero row-sums

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$$L\mathbb{1}_n = \mathbb{0}_n.$$

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A matrix $L \in \mathbb{R}^{n\times n}$, $n \geq 2$, is Laplacian if

1. its row-sums are zero,
2. its non-diagonal entries are non-positive, and
3. its diagonal entries are non-negative.

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Zero column-sums

Let G be a weighted digraph with Laplacian L and n nodes. Then

1. G is weight-balanced, i.e., $D_{\text{out}} = D_{\text{in}}$, **if and only if**
2. $\mathbb{1}_n^\top L = \mathbb{0}_n^\top$.

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Semisimplicity of the zero eigenvalue

Let L be the Laplacian matrix of a weighted digraph G with n nodes. Let n_s denote the number of sinks in the condensation digraph of G. Then

1. the eigenvalue 0 is semisimple with multiplicity n_s,
2. the following are equivalent:
   a. G contains a globally reachable node,
   b. the eigenvalue 0 is simple, and
   c. $\mathrm{rank}(L) = n - 1$

Today

Laplacian eigenvalues and algebraic connectivity

Suppose the Laplacian L is symmetric, then the eigenvalues are

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Algebraic connectivity

The second smallest eigenvalue λ_2 of a symmetric Laplacian L of a weighted digraph G is called the algebraic connectivity of G.

The algebraic connectivity and its associated eigenvector are also referred to as the Fiedler eigenvalue and Fiedler eigenvector.

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Algebraic connectivity and connectivity

For a weighted undirected graph G with symmetric Laplacian L:

1. G is connected if and only if $\lambda_2 > 0$;
2. the multiplicity of 0 is equal to the number of connected components.

Properties of the algebraic connectivity

Properties of algebraic connectivity

Consider a weighted undirected graph with symmetric adjacency matrix A, symmetric Laplacian matrix L, and algebraic connectivity λ_2.

1. Variational description:

$$\lambda_2 = \min_{\|x\|_2=1,\, x\perp \mathbb{1}_n} x^\top L x,$$

2. Monotonicity property:

$$A \leq A' \quad \Longrightarrow \quad \lambda_2 \leq \lambda_2'$$

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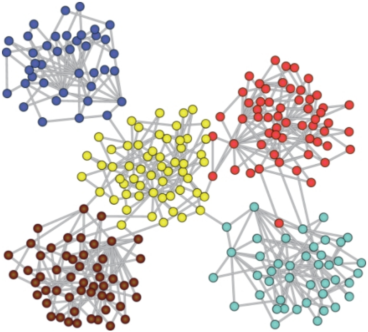
Graph	Algebraic connectivity
path graph P_n	$2(1-\cos(\pi/n)) \sim \pi^2/n^2$
cycle graph C_n	$2(1-\cos(2\pi/n)) \sim 4\pi^2/n^2$
star graph S_n	1
complete graph K_n	n
complete bipartite $K_{n,m}$	$\min(n,m)$

Community detection via algebraic connectivity



- Given adjacency matrix A, find a partition $V = V_1 \cup V_2$ to minimize

$$J(V_1, V_2) = \sum_{i \in V_1, j \in V_2} a_{ij} .$$

- Define $x \in \{-1, 1\}^n$ so that $x_i = 1$ iff $i \in V_1$, $x_i = -1$ iff $i \in V_2$

$$J(x) = \frac{1}{8} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2 = \frac{1}{4} x^\top L x$$

Community detection via algebraic connectivity

- Minimization of cut size

$$\underset{x\in\{-1,1\}^n\setminus\{-\mathbb{1}_n,\mathbb{1}_n\}}{\text{minimize}} \quad x^\top Lx.$$

- Continuous relaxation

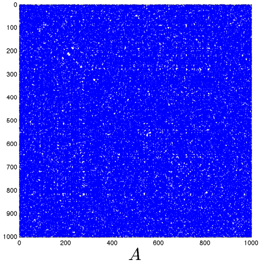
$$\underset{y\in\mathbb{R}^n,\, y\perp\mathbb{1}_n,\, \|y\|_\infty=1}{\text{minimize}} \quad y^\top Ly.$$

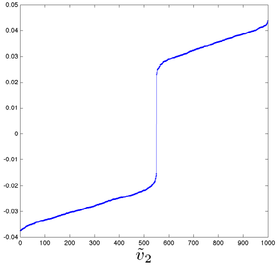
- Change norm

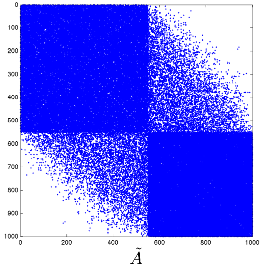
$$\underset{y\in\mathbb{R}^n,\, y\perp\mathbb{1}_n,\, \|y\|_2=1}{\text{minimize}} \quad y^\top Ly.$$

Heuristic: use sign of the Fiedler eigenvector to find a partition.

Community detection via algebraic connectivity



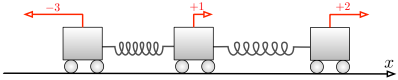


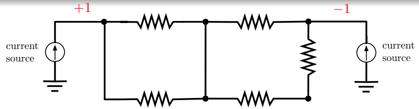


- $|V_1| = 450$, $|V_2| = 550$
- Nodes within V_1 are connected with probability 50%
- Nodes within V_2 are connected with probability 40%
- Nodes in V_1 and V_2 are connected with probability 15%

Today

Spring networks and resistive circuit





- Equilibrium displacement

$$L_{\text{stiffness}} x = f_{\text{load}}$$

- Voltage equilibrium

$$L_{\text{conductance}} v = c_{\text{injected}}$$

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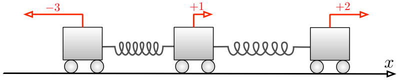

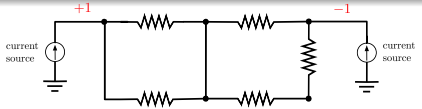

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Laplacian systems

A Laplacian system is a linear system of equations in the variable $x \in \mathbb{R}^n$ of the form

$$Lx = b,$$

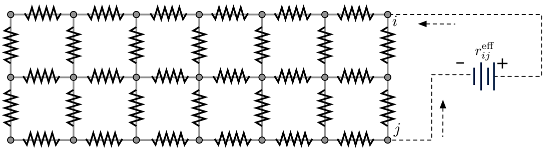
where $L \in \mathbb{R}^{n \times n}$ is a Laplacian matrix and $b \in \mathbb{R}^n$.

Soltuions to Laplacian systems

Solutions to Laplacian systems

Consider the symmetric Laplacian matrix L of a connected graph with decomposition $L = U \operatorname{diag}(0, \lambda_2, \ldots, \lambda_n) U^\top$, where $U \in \mathbb{R}^{n \times n}$ is orthonormal. Then

1. $\operatorname{image}(L) = \mathbb{1}_n^\perp$ so that the system admits solutions iff $b \perp \mathbb{1}_n$,
2. if $b \in \mathbb{R}^n$ is balanced, that is, $b \perp \mathbb{1}_n$, then the set of solutions to the Laplacian system is
$$\{L^\dagger b + \beta \mathbb{1}_n \mid \beta \in \mathbb{R}\},$$
3. the pseudoinverse of L is
$$L^\dagger = U \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 1/\lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1/\lambda_n \end{bmatrix} U^\top$$



- Injected current is $c_{\text{injected}} = \mathbb{e}_i - \mathbb{e}_j$
- Node potential v can be solved via

$$L_{\text{conductance}} v = c_{\text{injected}} \implies v = L^{\dagger}_{\text{conductance}} c_{\text{injected}} + \beta \mathbb{1}_n$$

- Effective resistance is then $r_{ij}^{\text{eff}} = v_i - v_j$, i.e.,

$$r_{ij}^{\text{eff}} = (\mathbb{e}_i - \mathbb{e}_j)^\top L^{\dagger}_{\text{conductance}} (\mathbb{e}_i - \mathbb{e}_j).$$

Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation