Continuous-time Averaging Systems

AU7036: Introduction to Multi-agent Systems

Xiaoming Duan
Department of Automation
Shanghai Jiao Tong University

March 19, 2024

Last time

- Definitions of Laplacian matrices
- Properties of Laplacian matrices
  1. Spectrum
  2. Multiplicity of eigenvalue 0 v.s. graph properties
- Symmetric Laplacian and algebraic connectivity
  - Multiplicity of eigenvalue 0 and number of connected components
  - Algebraic connectivity (Fiedler eigenvalue and Fiedler eigenvector)
- Laplacian systems and pseudoinverses

Today

1. Example systems
2. Continuous-time linear systems and their convergence properties
3. The Laplacian flow
4. Design of weight-balanced digraphs

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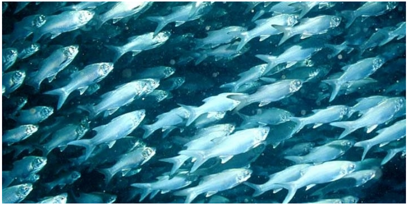

Flocking dynamics: a simple alignment rule

- Each animal steers towards the average heading of its neighbors

$$\dot{\theta}_i = \begin{cases} (\theta_j - \theta_i), & \text{if one neighbor} \\ \frac{1}{2}(\theta_{j_1} - \theta_i) + \frac{1}{2}(\theta_{j_2} - \theta_i), & \text{if two neighbors} \\ \frac{1}{m}(\theta_{j_1} - \theta_i) + \cdots + \frac{1}{m}(\theta_{j_m} - \theta_i), & \text{if } m \text{ neighbors} \end{cases}$$

$$= \text{average}\big(\{\theta_j,\ \text{for all neighbors } j\}\big) - \theta_i$$

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$$= \text{average}\big(\{\theta_j, \text{ for all neighbors } j\}\big) - \theta_i$$

In matrix form

$$\dot\theta = A\theta - \theta = (A - I)\theta = -L\theta$$

Example #2: Continuous-time opinion dynamics

- Discrete-time opinion dynamics

$$x(k+1) = Ax(k)$$

- A continuous-time counterpart

$$\begin{aligned} x_i(k+1) &= \sum_{j=1}^{n} a_{ij} x_j \\ &= a_{ii} x_i(k) + \sum_{j \neq i} a_{ij} x_j(k) \\ &= (1 - \sum_{j \neq i} a_{ij}) x_i(k) + \sum_{j \neq i} a_{ij} x_j(k) \\ &= x_i(k) + \sum_{j \neq i} a_{ij} (x_j(k) - x_i(k)) \end{aligned}$$

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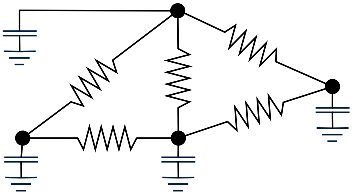
- If the edge weights a_{ij} are of the form $a_{ij} = \bar{a}_{ij}\tau$

$$\dot{x}(t) = -\bar{L}x(t)$$

Example #3: A simple RC circuit



- The injected currents at nodes satisfy

$$c_{\text{injected}} = L\,v,$$

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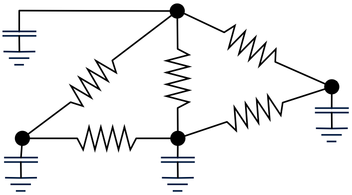


- The injected currents at nodes satisfy

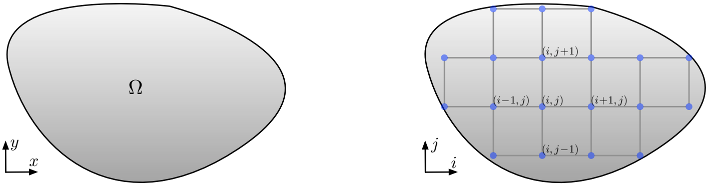
$$c_{\text{injected}} = L\,v,$$

- With capacitors at node i

$$C_i \frac{d}{dt} v_i = -c_{\text{injected at } i}$$

- The overall dynamics

$$\frac{d}{dt} v = -C^{-1} L\,v$$



- Let $u(t, x, y)$ be temperature at $(x, y) \in \Omega$ at time t, heat equation

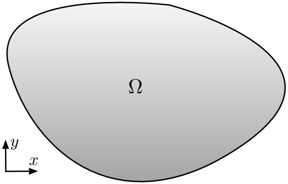
$$\frac{\partial u}{\partial t} = c\,\Delta u,$$

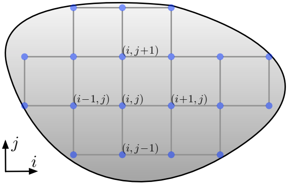
where

1. c is thermal diffusivity
2. Δ is Laplacian differential operator

$$\Delta u(t, x, y) = \frac{\partial^2 u}{\partial x^2}(t, x, y) + \frac{\partial^2 u}{\partial y^2}(t, x, y)$$

Example #4: Discretization of partial differential equations





- Finite difference approximation for Laplacian operator

$$\Delta u(t, x_i, y_j)$$
$$\approx \frac{u(t, x_{i-1}, y_j) + u(t, x_{i+1}, y_j) + u(t, x_i, y_{j-1}) + u(t, x_i, y_{j+1}) - 4u(t, x_i, y_j)}{h^2}$$

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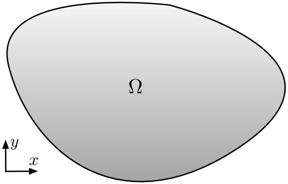

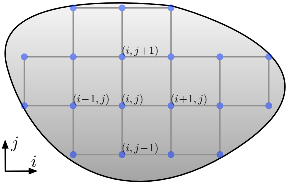

- Finite difference approximation for Laplacian operator

$$\Delta u(t, x_i, y_j) \approx \frac{u(t, x_{i-1}, y_j) + u(t, x_{i+1}, y_j) + u(t, x_i, y_{j-1}) + u(t, x_i, y_{j+1}) - 4u(t, x_i, y_j)}{h^2}$$

- Heat equation can be approximated by

$$\frac{d}{dt} u_{\text{discrete}} = -\frac{c}{h^2} L\, u_{\text{discrete}}$$

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3. The Laplacian flow
4. Design of weight-balanced digraphs

Continuous-time linear systems

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$$x(t) = \exp(At)x(0)$$

where

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

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Semi-convergence and convergence

A matrix $A \in \mathbb{R}^{n \times n}$ is

1. continuous-time semi-convergent if $\lim_{t \to +\infty} \exp(At)$ exists;
2. Hurwitz or continuous-time convergent if $\lim_{t \to +\infty} \exp(At) = \mathbb{0}_{n \times n}$

Convergence conditions

$$\dot{x}(t) = Ax(t)$$

- The spectral abscissa of A is maximum of real parts of eigenvalues:

$$\alpha(A) = \max\{\Re(\lambda) \mid \lambda \in \mathrm{spec}(A)\}$$

Convergence and spectral abscissa

For a square matrix A, the following statements hold:

1. A is continuous-time convergent (Hurwitz) if and only if $\alpha(A) < 0$,

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Convergence and spectral abscissa

For a square matrix A, the following statements hold:

1. A is continuous-time convergent (Hurwitz) if and only if $\alpha(A) < 0$,
2. A is continuous-time semi-convergent and not convergent iff
   a. 0 is a semisimple eigenvalue;
   b. all other eigenvalues have negative real part.

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Laplacian flow: Definition

Let G be a weighted digraph with n nodes and Laplacian matrix L

- The Laplacian flow on $\mathbb{R}^n$ is the dynamical system

$$\dot{x} = -Lx$$

- In components

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_j - x_i)$$

$$\dot{x} = -Lx$$

- The solution to the Laplacian flow is given by

$$x(t) = \exp(-Lt)x(0)$$

The matrix exponential of a Laplacian matrix

Let $L \in \mathbb{R}^{n \times n}$ be a Laplacian matrix with associated weighted digraph G and with maximum diagonal entry $\ell_{\max} = \max\{\ell_{11}, \ldots, \ell_{nn}\}$. Then

1. $\exp(-L)\mathbb{1}_n = \mathbb{1}_n$,

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3. $\exp(-L) \geq e^{-\ell_{\max}} I_n \geq \mathbb{0}_{n \times n}$,
4. $\exp(-L)\mathbb{e}_j > 0$, iff j-th node is globally reachable in G,

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5. $\exp(-L) > 0$, iff G strongly connected (L irreducible).

Equilibria and convergence of the Laplacian flow

$$\dot{x} = -Lx$$

Equilibria

If G contains a globally reachable node, then the set of equilibria is

$$\text{span}\{\mathbb{1}_n\} = \{\beta \mathbb{1}_n \mid \beta \in \mathbb{R}\}$$

Equilibria and convergence of the Laplacian flow

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Consensus conditions for $x(k+1) = Ax(k)$

Let A be a row-stochastic matrix and let G be its associated digraph. The following statements are equivalent:

(A1) the eigenvalue 1 is simple and all other eigenvalues μ satisfy $|\mu| < 1$;

(A2) A is semi-convergent and $\lim_{k\to\infty} A^k = \mathbb{1}_n w^\top$, where $w \geq 0$, $w^\top A = w^\top$ and $w^\top \mathbb{1}_n = 1$;

(A3) G contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic.

Equilibria and convergence of the Laplacian flow

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Let L be a Laplacian matrix and let G be its associated digraph. The following statements are equivalent:

(A1) the eigenvalue 0 of $-L$ is simple and all others have negative real part,

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Equilibria and convergence of the Laplacian flow

Consensus for Laplacian matrices with a globally reachable node

Let L be a Laplacian matrix and let G be its associated digraph. If any of (A1)-(A3) holds, then

1. $w \geq 0$ is left dominant eigenvector of $-L$ and $w_i > 0$ iff node i is globally reachable;

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3. if additionally G is weight-balanced, then G is strongly connected, $\mathbb{1}_n^\top L = \mathbb{0}_n^\top$, $w = \frac{1}{n}\mathbb{1}_n$, and

$$\lim_{t\to\infty} x(t) = \frac{\mathbb{1}_n^\top x(0)}{n}\mathbb{1}_n = \text{average}\big(x(0)\big)\mathbb{1}_n.$$

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Design of weight-balanced digraphs

Given a strongly-connected weighted digraph G with adjacency matrix A, how to design doubly stochastic $\bar{A}$ and weight balanced $\bar{L}$?

- Let $\ell_{\max} = \max\{\ell_{11}, \ldots, \ell_{nn}\}$, w be left dominant eigenvector of L
- Weight balanced $\bar{L}$

$$\bar{L} = \frac{1}{\ell_{\max}} \operatorname{diag}(w) L$$

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$$\bar{L} = \frac{1}{\ell_{\max}} \operatorname{diag}(w) L$$

- Doubly stochastic $\bar{A}$

$$\bar{A} = I_n - \bar{L}$$

Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems
- (*) The incidence matrix and its applications
- (*) Metzler matrices and dynamical flow systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation