Diffusively-Coupled Linear Systems

AU7036: Introduction to Multi-agent Systems

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Last time

- Continuous-time linear systems and their convergence properties
  - Solutions via matrix exponential
  - Semi-convergence and Hurwitz via spectral abscissa
- Laplacian flows and their convergence properties
  - Properties of matrix exponential of Laplacian matrices
  - Equivalent conditions for consensus
- Design of weight-balanced digraphs

Today

1. Consensus problems with vector-valued states
2. Diffusively-coupled linear systems (SISO case)
3. Application to the second-order Laplacian flow
4. State-feedback and controller design (MIMO case)
5. Extension to Formation Control

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Let G be a weighted digraph with n nodes and Laplacian matrix L

- The Laplacian flow on $\mathbb{R}^n$ is the dynamical system

$$\dot{x} = -Lx$$

- In components

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_j - x_i)$$

Laplacian flow with scalar states

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What if each node has a vector-valued state $x_i \in \mathbb{R}^d$?

- Position of robots in 2D or 3D space
- Opinions regarding multiple issues/topics

Vector-value states Laplacian flow

Component-wise equation still holds

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_j - x_i)$$

where $x_i \in \mathbb{R}^d$

Vector-value states Laplacian flow

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- Define system state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{nd}$$

- What is the differential equation for x?

Laplacian flow with vector-value states

- Define system state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{nd}$$

- What is the differential equation for x?

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -\ell_{11} I_d & -\ell_{12} I_d & \cdots & -\ell_{1n} I_d \\ -\ell_{21} I_d & -\ell_{22} I_d & \cdots & -\ell_{2n} I_d \\ \vdots & \vdots & \ldots & \vdots \\ -\ell_{n1} I_d & -\ell_{n2} I_d & \cdots & -\ell_{nn} I_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Laplacian flow with vector-value states

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- More compactly via Kronecker product

$$\dot{x} = (-L \otimes I_d)x$$

Kronecker product: Definition

Kronecker product

The Kronecker product of $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{q \times r}$ is the $nq \times mr$ matrix $A \otimes B$ given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}.$$

- What is $I_n \otimes B$?
- What is $A \otimes I_q$?
- What is $v \otimes w$?

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- What is $I_n \otimes B$?
- What is $A \otimes I_q$?
- What is $v \otimes w$?

$$I_n \otimes B = \begin{bmatrix} B & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & B \end{bmatrix}_{nq\times nr} \quad A \otimes I_q = \begin{bmatrix} a_{11}I_q & \dots & a_{1n}I_q \\ \vdots & \ddots & \vdots \\ a_{m1}I_q & \ddots & a_{mn}I_q \end{bmatrix}_{mq\times nq} \quad v \otimes w = \begin{bmatrix} v_1 w \\ \vdots \\ v_n w \end{bmatrix}$$

Kronecker product: Properties

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}$$

- Bilinearity

$$(\alpha A + \beta B) \otimes (\gamma C + \delta D) = \alpha\gamma A \otimes C + \alpha\delta A \otimes D + \beta\gamma B \otimes C + \beta\delta B \otimes D$$

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$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

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- Transpose

$$(A \otimes B)^\top = A^\top \otimes B^\top$$

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- Mixed product

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

Kronecker product: Consequences of mixed product

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ddots & a_{nm}B \end{bmatrix}$$

Mixed product property: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- If $Av = \lambda v$, $Bw = \mu w$, then

$$(A \otimes B)(v \otimes w) = (Av) \otimes (Bw) = \lambda\mu(v \otimes w)$$

  1. $v \otimes w$ is an eigenvector of $A \otimes B$
  2. $\text{spectrum}(A \otimes B) = \{\lambda\mu \mid \lambda \in \text{spectrum}(A), \mu \in \text{spectrum}(B)\}$

- If both A and B are invertible, then

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = I$$

$$\dot{x} = (-L \otimes I_d)x$$

or component-wise

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$$

- Decomposition: each component x_i^k of x_i for $k \in \{1, \ldots, d\}$ satisfies

$$\dot{x}_i^k = \sum_{j=1}^{n} a_{ij}(x_j^k - x_i^k)$$

  Component k reaches consensus to $w^\top \begin{bmatrix} x_1^k(0) & x_2^k(0) & \cdots & x_n^k(0) \end{bmatrix}^\top$
- Use the spectrum properties of $-L \otimes I_d$

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Diffusively-coupled identical linear systems

- A single-input-single-output (SISO) continuous-time linear system is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$
$$y_i(t) = Cx_i(t)$$

  where $x_i(t) \in \mathbb{R}^d$, $u_i(t) \in \mathbb{R}$, and $y_i(t) \in \mathbb{R}$
- Agents are connected through a weighted digraph G with Laplacian L

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- Output-dependent diffusive coupling law

$$u_i(t) = \sum_{j=1}^{n} a_{ij}(y_j(t) - y_i(t)) = C\sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t))$$

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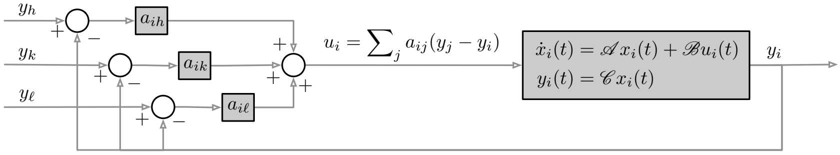

Diffusively-coupled identical linear systems

- A diffusively coupled SISO linear systems

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Diffusively-coupled identical linear systems

A network of diffusively-coupled identical linear systems is composed by n identical continuous-time linear SISO systems (A, B, C) and a Laplacian L.

Diffusively-coupled identical linear systems

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Asymptotic synchronization

A network of diffusively-coupled linear systems (A, B, C) and Laplacian L achieves asymptotic synchronization if, for all i, j and initial conditions,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\|_2 = 0.$$

1. There exists a trajectory $x_0(t)$ such that for all $i \in \{1, \ldots, n\}$

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\|_2 = 0$$

2. Synchronization to some trajectory $x_0(t)$ instead of a fixed value

Modeling via Kronecker product

- A diffusively coupled SISO linear systems

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- Closed-loop system

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$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ddots & A \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} -\sum_{j\neq 1} a_{1j}BC & \dots & a_{1n}BC \\ \vdots & \ddots & \vdots \\ a_{n1}BC & \ddots & -\sum_{j\neq n} a_{nj}BC \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

- What is Kronecker formulation with state $x = \begin{bmatrix} x_1^\top & x_2^\top & \dots & x_n^\top \end{bmatrix}^\top$?

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$$\dot{x} = (I_n \otimes A - L \otimes BC)x$$

Synchronization theorem

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Synchronization of diffusively-coupled linear systems

Consider a network of diffusively-coupled identical linear systems described by the system (A, B, C) and the Laplacian L.
Suppose the digraph associated with L contains a globally reachable node.
Then:

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- system synchronizes on $\bar{x}(t)$ iff $A - \lambda_i BC$ is Hurwitz for $2 \leq i \leq n$:
  1. Laplacian spectrum $0 = \lambda_1 < |\lambda_2| \leq \cdots \leq |\lambda_n|$
  2. Consensus value:
  $$\bar{x}(t) = e^{At} \sum_{i=1}^{n} w_i x_i(0)$$
  with $w^\top L = 0_n^\top$, $w^\top \mathbb{1}_n = 1$

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  with $w^\top L = 0_n^\top$, $w^\top \mathbb{1}_n = 1$
- system is exponentially stable iff $A - \lambda_i BC$ is Hurwitz for $1 \leq i \leq n$

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Second-order Laplacian flow: Definition

Let G be a weighted digraph with n nodes and Laplacian matrix L

- The Laplacian flow on $\mathbb{R}^n$ is the dynamical system for $i \in \{1, \dots, n\}$

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_j - x_i)$$

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- A double integrator is a dynamical system

$$\ddot{q}_i = \bar{u}_i, \qquad \text{or in first-order equivalent form} \begin{cases} \dot{q}_i = v_i \\ \dot{v}_i = \bar{u}_i \end{cases}$$

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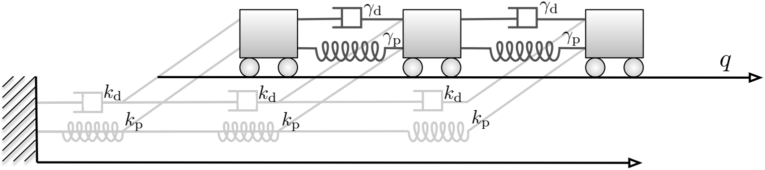
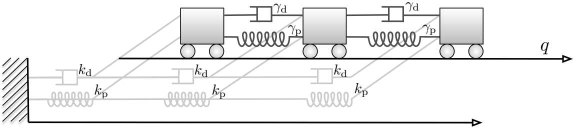


$$\bar{u}_i = -k_p q_i - k_d \dot{q}_i + \sum_{j=1}^{n} a_{ij}\big(\gamma_p(q_j - q_i) + \gamma_q(\dot{q}_j - \dot{q}_i)\big)$$

Second-order Laplacian flow: Definition



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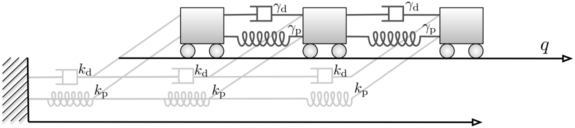
- The second-order Laplacian flow is given by

$$\ddot{q}(t) + (k_d I_n + \gamma_d L)\dot{q}(t) + (k_p I_n + \gamma_d L) q(t) = \mathbb{0}_n.$$



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- Let the state be

$$x_i = \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix}$$

- The dynamics of the i-th subsystem is

$$\dot{x}_i = \begin{bmatrix} \dot{q}_i \\ \ddot{q}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i$$

$$y_i = \begin{bmatrix} \gamma_p & \gamma_d \end{bmatrix} \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix}$$

Second-order Laplacian flow: Analysis

Synchronization theorem

The system achieves asymptotic synchronization on $\bar{x}(t)$ if and only if each matrix $A - \lambda_i BC$ for $i \in \{2, \dots, n\}$ is Hurwitz, where

1. Laplacian spectrum $0 = \lambda_1 < |\lambda_2| \leq \cdots \leq |\lambda_n|$
2. Consensus value: for $w^\top L = 0_n^\top$, $w^\top \mathbb{1}_n = 1$

$$\bar{x}(t) = e^{At} \sum_{i=1}^{n} w_i x_i(0)$$

$$\dot{x}_i = \begin{bmatrix} \dot{q}_i \\ \ddot{q}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_\mathrm{p} & -k_\mathrm{d} \end{bmatrix} \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \qquad y_i = \begin{bmatrix} \gamma_\mathrm{p} & \gamma_\mathrm{d} \end{bmatrix} \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix}$$

- Synchronization condition: the following matrix is Hurwitz

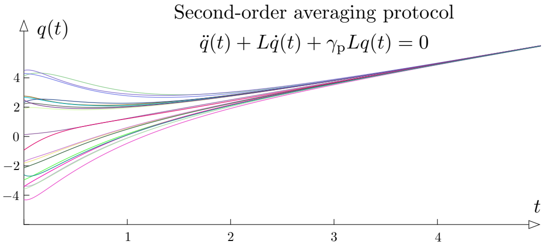
$$\begin{bmatrix} 0 & 1 \\ -k_\mathrm{p} & -k_\mathrm{d} \end{bmatrix} - \lambda_i \begin{bmatrix} 0 & 0 \\ \gamma_\mathrm{p} & \gamma_\mathrm{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(k_\mathrm{p} + \lambda_i \gamma_\mathrm{p}) & -(k_\mathrm{d} + \lambda_i \gamma_\mathrm{d}) \end{bmatrix}$$

- Consensus value:

$$\bar{x}(t) = e^{At}\sum_{i=1}^{n} w_i x_i(0) = e^{\begin{bmatrix} 0 & 1 \\ -k_\mathrm{p} & -k_\mathrm{d} \end{bmatrix} t}\sum_{i=1}^{n} w_i x_i(0)$$

1. $k_\mathrm{p} = k_\mathrm{d} = 0$ (no spring and no damper to wall, appropriate γ_p, γ_d):

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \qquad \begin{cases} q_i(t) \to t\sum_{i=1}^{n} w_i \dot{q}_i(0) + \sum_{i=1}^{n} w_i q_i(0) \\ \dot{q}_i(t) \to \sum_{i=1}^{n} w_i \dot{q}_i(0) \end{cases}$$



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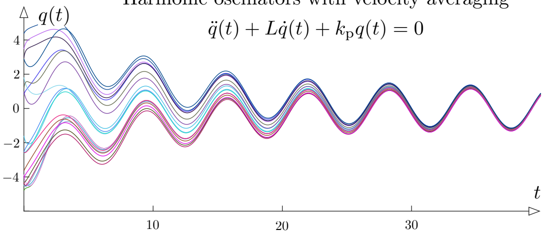
2. $k_\mathrm{p} > 0$, $k_\mathrm{d} = 0$ (only spring to wall, appropriate γ_p, γ_d):

$$q_i(t) \to \sum_{i=1}^{n} w_i q_i(0)\cos(\sqrt{k_\mathrm{p}}t) + \frac{1}{\sqrt{k_\mathrm{p}}}\sum_{i=1}^{n} w_i \dot{q}_i(0)\sin(\sqrt{k_\mathrm{p}}t);$$

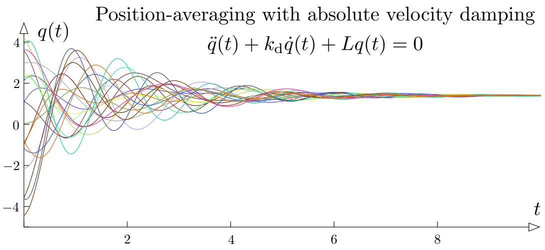


- Consensus value:

$$\bar{x}(t) = e^{At}\sum_{i=1}^{n} w_i x_i(0) = e^{\begin{bmatrix} 0 & 1 \\ -k_\mathrm{p} & -k_\mathrm{d} \end{bmatrix} t} \sum_{i=1}^{n} w_i x_i(0)$$

3. $k_\mathrm{p} = 0,\ k_\mathrm{d} > 0$ (only damper to wall, appropriate γ_p, γ_d):

$$e^{At} = \begin{bmatrix} 1 & \frac{1}{k_\mathrm{d}}(1 - e^{-k_\mathrm{d} t}) \\ 0 & e^{-k_\mathrm{d} t} \end{bmatrix} \qquad \begin{cases} q_i(t) \to \frac{\sum_{i=1}^{n} w_i \dot{q}_i(0)}{k_\mathrm{d}} + \sum_{i=1}^{n} w_i q_i(0) \\ \dot{q}_i(t) \to 0 \end{cases}$$



Today

Diffusively-coupled MIMO systems

- A multi-input-multi-output (MIMO) continuous-time linear system is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$
$$y_i(t) = Cx_i(t)$$

where $x_i(t) \in \mathbb{R}^d$, $u_i(t) \in \mathbb{R}^p$, and $y_i(t) \in \mathbb{R}^q$

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- State-dependent diffusive coupling law

$$u_i(t) = c\sum_{j=1}^{n} a_{ij}K(x_j(t) - x_i(t))$$

where $c > 0$ is coupling coefficient, $K \in \mathbb{R}^{p\times d}$ is control gain matrix

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How to design c and K so as to achieve synchronization?

Diffusively-coupled MIMO systems

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Synchronization theorem

Consider a network of state-feedback diffusively-coupled identical linear systems described by the system (A, B, C) and the Laplacian L.
Suppose the digraph associated with L contains a globally reachable node.
Then:

- system achieves synchronization on $\bar{x}(t)$ iff each matrix $A - c\lambda_i BK$ is Hurwitz for $2 \leq i \leq n$, where with $w^\top L = 0_n^\top$, $w^\top \mathbb{1}_n = 1$

$$\bar{x}(t) = e^{At}\sum_{i=1}^{n} w_i x_i(0)$$

Stabilization of linear systems

- A continuous-time linear control systems is

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times p}$

- Given feedback gain K, feedback $u = -Kx$ induces $\dot{x} = (A - BK)x$

Stabilizability

(A, B) is stabilizable if there exists matrix K such that $A - BK$ is Hurwitz.

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Stabilizability of linear control systems

Given a linear control system (A, B), the following are equivalent

1. (A, B) is stabilizable;
2. there exists $d \times d$ matrix $P \succ 0$ solving the Lyapunov inequality
$$AP + PA^\top - 2BB^\top \prec 0 \qquad \text{linear matrix inequality (LMI)}$$

A stabilizing feedback gain matrix is $K = B^\top P^{-1}$.

Diffusively-coupled MIMO systems: Analysis

Algorithm High-gain LMI design

Input: a stabilizable pair (A, B)

Output: a control gain matrix K and coupling gain c

1: set $P :=$ any solution to the LMI $AP + PA^\top - 2BB^\top \prec 0$

2: set $K := B^\top P^{-1}$

3: set $c := 1/\min\{\Re(\lambda_i) \mid i \in \{2, \ldots, n\}\}$

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Lyapunov inequality for Hurwitzness

A complex matrix $A \in \mathbb{C}^{n \times n}$ is Hurwitz if there exists $P \succ 0$ such that

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High-gain LMI design for stabilizable linear control systems

Consider n identical continuous-time linear control systems described by (A, B) and a digraph G with Laplacian L.

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High-gain LMI design for stabilizable linear control systems

Consider n identical continuous-time linear control systems described by (A, B) and a digraph G with Laplacian L.
If (A, B) stabilizable and G contains a globally reachable node, then the resulting pair (K, c) of the high-gain LMI design algorithm ensures that each matrix $A - c\lambda_i BK$ is Hurwitz for $2 \le i \le n$.

Today

1. Consensus problems with vector-valued states
2. Diffusively-coupled linear systems (SISO case)
3. Application to the second-order Laplacian flow
4. State-feedback and controller design (MIMO case)
5. Extension to Formation Control

Formation control of multi-agent systems

- A multi-input-multi-output (MIMO) continuous-time linear system is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$
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where $x_i(t) \in \mathbb{R}^d$, $u_i(t) \in \mathbb{R}^p$, and $y_i(t) \in \mathbb{R}^q$

- Let $H = (h_1, \ldots, h_n)$ be a constant formation of agent network

How to design controller u_i so that agents achieve formation?

Multi-agent formation

A network of linear systems (A, B, C) with Laplacian L achieves formation $H = (h_1, \ldots, h_n)$ if, for all i, j and initial conditions,

$$\lim_{t \to \infty} \|(x_i(t) - h_i) - (x_j(t) - h_j)\|_2 = 0.$$

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- State-dependent formation control law

$$u_i(t) = c \sum_{j=1}^{n} a_{ij} K((x_j(t) - h_j) - (x_i(t) - h_i))$$

Formation control of multi-agent systems

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$$

$$u_i(t) = c\sum_{j=1}^{n} a_{ij}K((x_j(t) - h_j) - (x_i(t) - h_i))$$

- Closed-loop systems

$$\dot{x}_i(t) = Ax_i(t) + cBK\sum_{j=1}^{n} a_{ij}((x_j(t) - h_j) - (x_i(t) - h_i))$$

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- Closed-loop systems

$$\dot{x}_i(t) = Ax_i(t) + cBK\sum_{j=1}^{n} a_{ij}((x_j(t) - h_j) - (x_i(t) - h_i))$$

- Define new variable $\tilde{x}_i = x_i - h_i$, then we have

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + cBK\sum_{j=1}^{n} a_{ij}(\tilde{x}_j(t) - \tilde{x}_i(t)) + Ah_i$$

Agents achieve formation iff $\tilde{x}$ achieves consensus

Formation control of multi-agent systems

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + cBK\sum_{j=1}^{n} a_{ij}(\tilde{x}_j(t) - \tilde{x}_i(t)) + Ah_i$$

- Modeling using Kronecker product

$$\dot{\tilde{x}} = (I_n \otimes A - cL \otimes BK)\tilde{x} + (I_n \otimes A)h$$

where $h = \left[h_1^\top, \ldots, h_n^\top\right]^\top$

Formation control of multi-agent systems

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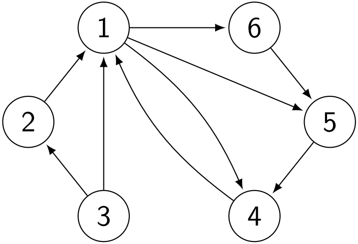
- the system achieves asymptotic formation iff each matrix $A - c\lambda_i BK$ is Hurwitz for $2 \leq i \leq n$, and $Ah_i = 0$ for $1 \leq i \leq n$.



- Each agent moves on a 2D plane with state $x_i = \begin{bmatrix} x_i^1 & x_i^2 & v_i^1 & v_i^2 \end{bmatrix}^\top$
- State space model

$$\dot{x}_i = \begin{bmatrix} \dot{x}_i^1 \\ \dot{x}_i^2 \\ \dot{v}_i^1 \\ \dot{v}_i^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \\ v_i^1 \\ v_i^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_i^1 \\ u_i^2 \end{bmatrix}$$

- Control for each agent

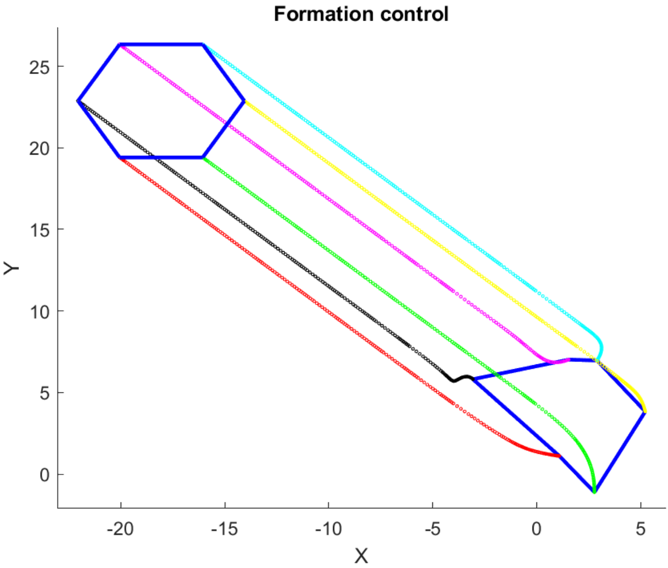
$$u_i = cK \sum_{j=1}^{n} a_{ij}((x_j(t) - h_j) - (x_i(t) - h_i))$$

where the formation is a regular hexagon (position formation):

$$h_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad h_3 = \begin{bmatrix} 6 \\ 2\sqrt{3} \\ 0 \\ 0 \end{bmatrix} \quad h_4 = \begin{bmatrix} 4 \\ 4\sqrt{3} \\ 0 \\ 0 \end{bmatrix} \quad h_5 = \begin{bmatrix} 0 \\ 4\sqrt{3} \\ 0 \\ 0 \end{bmatrix} \quad h_6 = \begin{bmatrix} -2 \\ 2\sqrt{3} \\ 0 \\ 0 \end{bmatrix}$$

- Note that $Ah_i = 0$ is satisfied for all i
- The gain matrix and coefficient K and c are designed using LMIs

Formation control of multi-agent systems: An example



Upcoming

Week 1-6:

- Introduction
- Elements of matrix theory
- Elements of graph theory
- Elements of algebraic graph theory
- Discrete-time averaging systems
- The Laplacian matrix
- Continuous-time averaging systems
- Diffusively-coupled linear systems

Week 7-14:

- Lyapunov stability theory
- Nonlienar averaging systems (Euler-Lagrangian, oscillators)
- Other advanced topics

Week 15-16:

- Project presentation